Université Pierre et Marie Curie – Paris 6 UFR 929 – Mathématiques

Quelques problèmes liés à la dynamique des équations de Gross–Pitaevskii et de Landau–Lifshitz

THÈSE

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Résumé

Cette thèse est consacrée à l'étude des équations de Gross–Pitaevskii et de Landau–Lifshitz, qui présentent d'importantes applications en physique. L'équation de Gross–Pitaevskii modélise des phénomènes de l'optique non linéaire, de la superfluidité et de la condensation de Bose–Einstein, tandis que l'équation de Landau–Lifshitz décrit la dynamique de l'aimantation dans des matériaux ferromagnétiques.

Lorsqu'on modélise la matière à très basse température, on fait l'hypothèse que l'interaction des particules est ponctuelle. L'équation de Gross–Pitaevskii classique s'en déduit alors en prenant comme interaction une masse de Dirac. Cependant, différents types de potentiels non locaux probablement plus réalistes ont aussi été proposés par des physiciens pour modéliser des interactions plus générales. Dans un premier temps, on s'intéressera à donner des conditions suffisantes couvrant une variété assez large d'interactions non locales et telles que le problème de Cauchy associé soit globalement bien posé avec des conditions non nulles à l'infini. Par la suite, on étudiera les ondes progressives de ce modèle non local et on donnera des conditions telles que l'on puisse déterminer les vitesses pour lesquelles il n'existe pas de solution non constante d'énergie finie.

Concernant l'équation de Landau–Lifshitz, on s'intéressera aussi aux ondes progressives d'énergie finie. On montrera la non existence d'ondes progressives non constantes d'énergie petite en dimensions deux, trois et quatre, sous l'hypothèse que l'énergie soit inférieure au moment dans le cas de la dimension deux. En outre, on donnera aussi dans le cas bidimensionnel la description d'une courbe minimisante qui pourrait donner une approche variationnelle pour construire des solutions de l'équation de Landau–Lifshitz. Finalement, on décrira le comportement à l'infini des ondes progressives d'énergie finie.

Mots-clés : Équation de Schrödinger non locale, Équation de Gross–Pitaevskii, Ondes progressives, Caractère globalement bien posé, Conditions non nulles à l'infini, Équation de Landau–Lifshitz, Applications harmoniques, Applications de Schrödinger.

Abstract

This thesis is devoted to the study of the Gross–Pitaevskii equation and the Landau–Lifshitz equation, which have important applications in physics. The Gross–Pitaevskii equation models phenomena of nonlinear optics, superfluidity and Bose–Einstein condensation, while the Landau–Lifshitz equation describes the dynamics of magnetization in ferromagnetic materials.

When modeling matter at very low temperatures, it is usual to suppose that the interaction between particles is punctual. Then the classical Gross–Pitaevskii equation is derived by taking as interaction the Dirac delta function. However, different types of nonlocal potentials, probably more realistic, have also been proposed by physicists to model more general interactions. First, we will focus on provide sufficient conditions that cover a broad variety of nonlocal interactions and such that the associated Cauchy problem is globally well-posed with nonzero conditions at infinity. After that, we will study the traveling waves for this nonlocal model and we will provide conditions such that we can compute a range of speeds in which nonconstant finite energy solutions do not exist.

Concerning the Landau–Lifshitz equation, we will also be interested in finite energy traveling waves. We will prove the nonexistence of nonconstant traveling waves with small energy in dimensions two, three and four, provided that the energy is less than the momentum in the two-dimensional case. In addition, we will also give, in the two-dimensional case, the description of a minimizing curve which could give a variational approach to build solutions of the Landau-Lifshitz equation. Finally, we describe the asymptotic behavior at infinity of the finite energy traveling waves.

Keywords: Nonlocal Schrödinger equation, Gross–Pitaevskii equation, Traveling waves, Global well-posedness, Nonzero conditions at infinity, Landau–Lifshitz equation, Harmonic maps, Schrödinger maps.

Chapitre 1

Introduction générale

1.1 L'équation de Gross–Pitaevskii non locale

1.1.1 Motivation physique

Afin de décrire la cinétique d'un gaz de Bose, dont les bosons de masse m interagissent faiblement, Gross [52] et Pitaevskii [86] ont trouvé, dans l'approximation de Hartree, que la fonction d'onde Ψ régissant le condensat vérifie l'équation

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2m}\Delta\Psi(x,t) + \Psi(x,t)\int_{\mathbb{R}^N}|\Psi(y,t)|^2 V(x-y)\,dy, \text{ dans } \mathbb{R}^N\times\mathbb{R}, \tag{1.1}$$

où N est la dimension spatiale et V décrit l'interaction entre les bosons. L'approximation la plus typique, où V est considéré comme une masse de Dirac, conduit à l'équation de Gross–Pitaevskii que l'on appellera l'équation de Gross–Pitaevskii classique ou locale (voir sous-section 1.1.2). Ce modèle local avec des conditions non nulles à l'infini a été intensivement utilisé en raison de son application à plusieurs domaines de la physique, comme la superfluidité, l'optique non linéaire et la condensation de Bose–Einstein [62, 61, 65, 26]. Il semble cependant naturel d'analyser l'équation (1.1) dans le cas d'interactions plus générales. En effet, dans l'étude de la superfluidité, des supersolides et de la condensation de Bose–Einstein, différents types de potentiels non locaux ont été proposés [6, 32, 96, 87, 63, 1, 103, 27, 23].

Pour obtenir une équation sans dimension, on prend le niveau moyen de l'énergie par l'unité de masse $\mathcal{E}_0$ d'un boson de masse m et l'on pose

$$\psi(x,t) = \exp\left(\frac{im\mathcal{E}_0 t}{\hbar}\right)\Psi(x,t).$$

Alors (1.1) prend la forme

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\Delta\psi(x,t) - m\mathcal{E}_0\psi(x,t) + \psi(x,t)\int_{\mathbb{R}^N}|\psi(y,t)|^2 V(x-y)\,dy. \tag{1.2}$$

En considérant le changement d'échelle, pour $\lambda > 0$ à choisir,

$$u(x,t) = \frac{1}{\lambda\sqrt{m\mathcal{E}_0}}\left(\frac{\hbar}{\sqrt{2m^2\mathcal{E}_0}}\right)^{\frac{N}{2}}\psi\left(\frac{\hbar x}{\sqrt{2m^2\mathcal{E}_0}}, \frac{\hbar t}{m\mathcal{E}_0}\right),$$

on déduit de (1.2) que

$$i\partial_t u(x,t) + \Delta u(x,t) + u(x,t)\left(1 - \lambda^2 \int_{\mathbb{R}^N} |u(y,t)|^2 \mathcal{V}(x-y)\,dy\right) = 0,$$

avec

$$\mathcal{V}(x) = V\left(\frac{\hbar x}{\sqrt{2m^2\mathcal{E}_0}}\right).$$

Si on admet que la convolution entre $\mathcal{V}$ et une constante est bien définie et égale à une constante positive, il est alors naturel de choisir

$$\lambda^2 = (\mathcal{V} * 1)^{-1}.$$

Avec ce choix, l'équation (1.2) est équivalente à

$$i\partial_t u + \Delta u + \lambda^2 u(\mathcal{V} * (1-|u|^2)) = 0 \text{ dans } \mathbb{R}^N \times \mathbb{R}. \tag{1.3}$$

Cette déduction conduit à considérer de manière plus générale l'équation de Gross–Pitaevskii non locale écrite sous la forme

$$i\partial_t u + \Delta u + u(W * (1-|u|^2)) = 0 \text{ dans } \mathbb{R}^N \times \mathbb{R}. \tag{GPN}$$

Si W est une distribution paire réelle, (GPN) est une équation Hamiltonienne et son énergie, donnée par

$$E_W(u(t)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t)|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N} (W * (1-|u(t)|^2))(1-|u(t)|^2)\,dx, \tag{1.4}$$

est formellement conservée. Si de plus on s'intéresse à des fonctions d'énergie finie, elles doivent avoir des conditions non nulles à l'infini. En particulier, on étudiera le problème de Cauchy pour l'équation (GPN) avec une donnée initiale $u(0) = u_0$ vérifiant

$$|u_0(x)| \to 1, \quad \text{lorsque} \quad |x| \to \infty. \tag{1.5}$$

On donne maintenant trois types de noyaux (non locaux) qui nous serviront pour illustrer nos résultats. D'abord, on considère le potentiel proposé par V. S. Shchesnovich et R. A. Kraenkel dans [96] pour $\varepsilon > 0$,

$$W_\varepsilon(x) = \begin{cases} \dfrac{1}{2\pi\varepsilon^2} K_0\left(\dfrac{|x|}{\varepsilon}\right), & N = 2, \\ \dfrac{1}{4\pi\varepsilon^2 |x|} \exp\left(-\dfrac{|x|}{\varepsilon}\right), & N = 3, \end{cases} \tag{1.6}$$

où K_0 est la fonction de Bessel modifiée de deuxième type (également appelée fonction de MacDonald). De cette façon W_ε pourrait être considéré comme une approximation de la masse de Dirac, puisque $W_\varepsilon \to \delta$, lorsque $\varepsilon \to 0$, au sens des distributions. Un deuxième exemple des interactions non locales est le *soft core potential*

$$1_{|x|\leq a}(x) = \begin{cases} 1, & \text{si } |x| < a, \\ 0, & \text{sinon}, \end{cases} \tag{1.7}$$

avec $a > 0$, qui est utilisé dans [63, 1] pour l'étude de supersolides. Finalement, on considère

$$W = a\delta + bK, \qquad a, b \in \mathbb{R}, \tag{1.8}$$

où K est le noyau singulier

$$K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}, \quad x \in \mathbb{R}^3 \backslash \{0\}. \tag{1.9}$$

Le potentiel (1.8)–(1.9) modélise les forces dipolaires dans un gaz quantique (voir [23], [103]).

Maintenant on raisonne de façon formelle et on considère une fonction constante u_0 de module égal à un. Comme (GPN) est invariant par un changement de phase, on peut supposer que $u_0 = 1$. Alors, l'équation linéarisée de (GPN) autour de u_0 est donnée par

$$i\partial_t \tilde{u} - \Delta \tilde{u} + 2W * \mathrm{Re}(\tilde{u}) = 0. \tag{1.10}$$

En écrivant $\tilde{u} = \tilde{u}_1 + i\tilde{u}_2$ et en prenant les parties réelle et imaginaire de (1.10), on a

$$\begin{aligned} -\partial_t \tilde{u}_2 - \Delta \tilde{u}_1 + 2W * \tilde{u}_1 &= 0, \\ \partial_t \tilde{u}_1 - \Delta \tilde{u}_2 &= 0, \end{aligned}$$

d'où

$$\partial_{tt}^2 \tilde{u} - 2W * (\Delta \tilde{u}) + \Delta^2 \tilde{u} = 0. \tag{1.11}$$

En imposant $\tilde{u} = e^{i(\xi.x - wt)}$, $w \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, comme une solution de (1.11), on obtient la relation de dispersion

$$(w(\xi))^2 = |\xi|^4 + 2\widehat{W}(\xi)|\xi|^2, \tag{1.12}$$

où $\widehat{W}$ désigne la transformée de Fourier de W :

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-ix.\xi}\, dx.$$

En supposant que $\widehat{W}$ est positif et continu à l'origine, on a dans le régime des ondes longues, i.e. $\xi \sim 0$, que

$$w(\xi) \sim (2\widehat{W}(0))^{1/2} |\xi|.$$

Par conséquent, dans ce régime, on peut identifier $(2\widehat{W}(0))^{1/2}$ à la vitesse des ondes sonores (aussi appelée la vitesse sonique), on pose donc

$$c_s(W) = (2\widehat{W}(0))^{1/2}. \tag{1.13}$$

La relation de dispersion (1.12) a été observée pour la première fois par Bogoliubov [14] dans l'étude d'un gaz de Bose–Einstein et sous certaines considérations physiques, il a établi que le gaz devait se déplacer à une vitesse inférieure à $c_s(W)$ pour préserver ses propriétés de superfluidité. Cette valeur sera fondamentale dans l'étude des ondes progressives pour cette équation.

1.1.2 Le problème de Cauchy pour l'équation de Gross–Pitaevskii

Comme nous l'avions signalé à la sous-section précédente, au cas où les interactions sont modélisées par un noyau W égal à une masse de Dirac, (GPN) prend la forme Gross–Pitaevskii classique

$$i\partial_t u + \Delta u + u(1 - |u|^2) = 0 \ \text{ dans } \ \mathbb{R}^N \times \mathbb{R}, \tag{GP}$$

On note alors que la fonctionnelle (1.4) correspond à l'énergie de Ginzburg–Landau

$$E_{GL}(u(t)) \equiv E_\delta(u(t)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t)|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N} (1-|u(t)|^2)^2\,dx,$$

qui est bien définie dans l'espace d'énergie

$$\mathcal{E}(\mathbb{R}^N) = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N),\ 1-|u|^2 \in L^2(\mathbb{R}^N)\}.$$

Comme on l'a déjà mentionné, on s'intéresse à des fonctions d'énergie finie qui imposent des conditions non nulles à l'infini. Donc il faut choisir des espaces bien adaptés à cette condition. Par exemple, on pourrait utiliser les espaces de Zhidkov :

$$X^k(\mathbb{R}^N) = \{u \in L^\infty(\mathbb{R}^N) : \nabla u \in H^{k-1}(\mathbb{R}^N)\}.$$

En effet, le premier résultat dans le cas unidimensionnel est le suivant.

Théorème 1 ([104, 105]). *Le problème de Cauchy pour l'équation* (GP) *est globalement bien posé dans l'espace* $X^1(\mathbb{R})$.

Quelques années plus tard, le théorème 1 a été étendu par C. Gallo (voir aussi O. Goubet)

Théorème 2 ([38, 45]). *Soit* $1 \le N \le 2$. *Alors le problème de Cauchy pour l'équation* (GP) *est globalement bien posé dans l'espace* $X^N(\mathbb{R}^N)$ *pour une donnée initiale* $u_0 \in \mathcal{E}(\mathbb{R}^N) \cap X^N(\mathbb{R}^N)$.

P. Gérard a étudié le problème directement dans $\mathcal{E}(\mathbb{R}^N)$, espace métrique complet pour la distance

$$d(u,v) = \|u-v\|_{X^1+H^1} + \||u|^2 - |v|^2\|_{L^2}.$$

Il montre le caractère bien posé en dimension $1 \le N \le 3$ et aussi en dimension $N = 4$ sous une condition de petitesse de la donnée initiale. Plus précisément,

Théorème 3 ([41, 40]). *Soit* $1 \le N \le 3$. *Pour tout* $u_0 \in \mathcal{E}(\mathbb{R}^N)$, *il existe une solution* $u \in C(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ *de* (GP) *de donnée initiale* $u(0) = u_0$. *De plus, pour tout* $R > 0$ *et tout* $T > 0$, *il existe* $C > 0$ *tel que pour tous* $u_0, \tilde{u}_0$ *tels que* $E_{GL}(u) \le R$ *et* $E_{GL}(\tilde{u}_0) \le R$, *les solutions correspondantes vérifient*

$$\sup_{|t|\le T} d(u(t), \tilde{u}(t)) \le C d(u_0, \tilde{u}_0).$$

En outre, pour $N = 4$, *il existe* $\delta > 0$ *tel que pour tout* $u_0 \in \mathcal{E}(\mathbb{R}^N)$ *vérifiant* $\mathcal{E}(u_0) \le \delta$, *il existe une solution* $u \in C(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ *de* (GP) *de donnée initiale* $u(0) = u_0$. *De plus, le caractère lipschitzien du flot énoncé ci-dessus est aussi vérifié.*

Une autre approche pour étudier ce problème est de travailler autour des états d'équilibre constants $\bar{u} = e^{i\theta}$, avec $\theta \in \mathbb{R}$. Puisque (GPN) est invariant par un changement de phase, on se réduit au cas $\bar{u} = 1$. Dans ce cadre, F. Béthuel et J.-C. Saut [12] ont établi

Théorème 4 ([12]). *Soit* $2 \le N \le 3$. *Alors le problème de Cauchy pour l'équation* (GP) *est globalement bien posé dans l'espace* $1 + H^1(\mathbb{R}^N)$.

Dans [39], C. Gallo a considéré une approche qui généralise le cadre du théorème 4 dans des espaces du type $\phi + H^1(\mathbb{R}^N)$, où ϕ est une fonction régulière d'énergie finie.

Théorème 5 ([39])**.** *Soit* $1 \leq N \leq 3$. *Alors pour toute fonction* ϕ *vérifiant*

$$\phi \in C^3(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \nabla\phi \in H^3(\mathbb{R}^N),\ |\phi|^2 - 1 \in L^2(\mathbb{R}^N),$$

l'équation (GP) *est globalement bien posée dans* $\phi + H^1(\mathbb{R}^N)$.

Finalement dans le cas $N \geq 4$, S. Gustafson et al. ([54]) ont étudié le comportement à l'infini des petites perturbations des états d'équilibre. En particulier, ils ont obtenu le caractère bien posé du problème, pour des donnés initiales petites.

Théorème 6 ([54])**.** *Soit* $N \geq 4$ *et* $s \geq N/2-1$. *Il existe* $\delta > 0$ *tel que pour toute donnée initiale* $u(0) = 1 + u_0$, *vérifiant* $\|u_0\|_{H^s} \leq \delta$, *le problème de Cauchy pour l'équation* (GP) *est globalement bien posé dans l'espace* $1+H^s(\mathbb{R}^N)$.

Une des questions clés que le présent travail tente de résoudre est la suivante : *quelles conditions doit-on imposer à* W *pour que le problème de Cauchy* (GPN) *soit globalement bien posé ?* Bien entendu, on cherche des conditions assez générales telles que la masse de Dirac et les potentiels (1.6), (1.7) et (1.8)-(1.9) soient inclus. Pour cette raison, on travaillera dans les espaces

$$\mathcal{M}_{p,q}(\mathbb{R}^N) = \{V \in S'(\mathbb{R}^N) : \exists C \geq 0 \text{ t.q. } \|V * f\|_{L^q} \leq C\|f\|_{L^p},\ \forall f \in L^p(\mathbb{R}^N)\},$$

i.e., les espaces de distributions tempérées V telles que l'opérateur linéaire $f \mapsto V * f$ est borné de $L^p(\mathbb{R}^N)$ dans $L^q(\mathbb{R}^N)$. On note par $\|V\|_{p,q}$ sa norme.

On supposera qu'il existe

$$p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4, s_1, s_2 \in [1, \infty),$$

en vérifiant

$$\frac{N}{N-2} > p_4, \quad \frac{2N}{N-2} > p_2, p_3, s_1, s_2 \geq 2, \quad 2 \geq q_1 > \frac{2N}{N+2}, \quad q_3, q_4 > \frac{N}{2} \quad \text{si } N \geq 3$$

et

$$p_2, p_3, s_1, s_2 \geq 2, \quad 2 \geq q_1 > 1 \quad \text{si } 2 \geq N \geq 1,$$

tels que

$$\begin{cases} W \in \mathcal{M}_{2,2}(\mathbb{R}^N) \cap \bigcap_{i=1}^{4} \mathcal{M}_{p_i,q_i}(\mathbb{R}^N), \\ \dfrac{1}{p_3} + \dfrac{1}{q_2} = \dfrac{1}{q_1}, \quad \dfrac{1}{p_1} - \dfrac{1}{p_3} = \dfrac{1}{s_1}, \quad \dfrac{1}{q_1} - \dfrac{1}{q_3} = \dfrac{1}{s_2} \quad \text{si } N \geq 3. \end{cases} \tag{$\mathcal{W}_N$}$$

On rappelle que si $p > q$, alors $\mathcal{M}_{p,q} = \{0\}$. Par conséquent, si l'on suppose que W n'est pas nulle, les nombres ci-dessus doivent satisfaire $q_2, q_3 \geq 2$. De plus, l'existence de s_1, s_2 et les relations en $(\mathcal{W}_N)$ impliquent

$$\frac{N}{N-2} > p_1, \quad q_2 > \frac{N}{2}, \quad \frac{1}{p_1} - \frac{1}{p_3} \in \left(\frac{N-2}{2N}, \frac{1}{2}\right], \quad \frac{1}{q_1} - \frac{1}{q_3} \in \left(\frac{N-2}{2N}, \frac{1}{2}\right] \quad \text{si } N \geq 3.$$

Dans le cas où $N \in \{1, 2, 3\}$, on peut choisir $(p_4, q_4) = (2, 2)$ dans $(\mathcal{W}_N)$. Donc la condition que $W \in \mathcal{M}_{p_4,q_4}(\mathbb{R}^N)$ n'est pas triviale seulement lorsque $N \geq 4$.

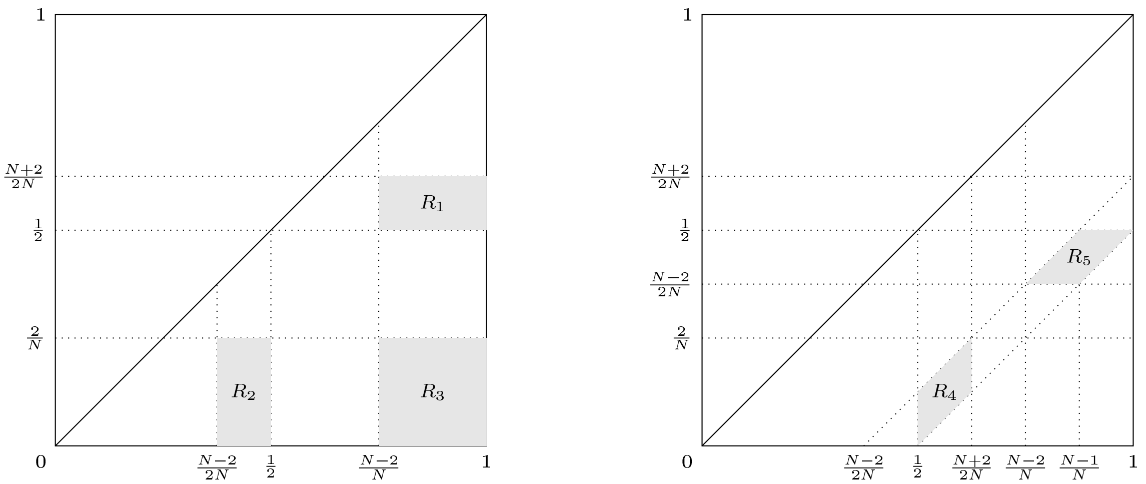

Figure 1.1 – *Pour $N > 4$, le plan $(1/p, 1/q)$ est représenté à gauche, dans le sens que $(1/p_1, 1/q_1) \in R_1$, $(1/p_2, 1/q_2), (1/p_3, 1/q_3) \in R_2$, $(1/p_4, 1/q_4) \in R_3$. À droite, les zones ombrées symbolisent le fait que $(1/q_1, 1/q_3) \in R_4$ et $(1/p_1, 1/p_3) \in R_5$, pour $N > 6$.*

La figure 1.1 montre schématiquement l'emplacement de ces nombres dans le carré unitaire.

On remarque que la condition $(\mathcal{W}_N)$ implique que $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$. En outre, puisque

$$W \in \mathcal{M}_{2,2}(\mathbb{R}^N) \iff \widehat{W} \in L^\infty(\mathbb{R}^N),$$

en utilisant l'identité de Plancherel, on en déduit que

$$E_W(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4(2\pi)^N}\int_{\mathbb{R}^N} \widehat{W}\,|\widehat{1-|v|^2}|^2 < +\infty, \tag{1.14}$$

pour tout $v \in \mathcal{E}(\mathbb{R}^N)$.

A première vue, il n'est pas évident de vérifier les hypothèses sur W. Le but de la proposition suivante consiste à donner des conditions suffisantes pour assurer $(\mathcal{W}_N)$.

Proposition 7.

(i) Soit $1 \leq N \leq 3$. Si $W \in \mathcal{M}_{2,2}(\mathbb{R}^N) \cap \mathcal{M}_{3,3}(\mathbb{R}^N)$, alors W satisfait $(\mathcal{W}_N)$. Par ailleurs, si W vérifie $(\mathcal{W}_N)$ avec $p_i = q_i$, $1 \leq i \leq 3$, alors $W \in \mathcal{M}_{2,2}(\mathbb{R}^N) \cap \mathcal{M}_{3,3}(\mathbb{R}^N)$.

(ii) Soit $N \geq 4$. On suppose que $W \in \mathcal{M}_{r,r}(\mathbb{R}^N)$ pour tout $1 < r < \infty$. De plus, s'il existe $\bar{r} > N/4$ tel que $W \in \mathcal{M}_{p,q}(\mathbb{R}^N)$, pour tout $1 - 1/\bar{r} < 1/p < 1$, où $1/q = 1/p + 1/\bar{r} - 1$, alors W vérifie $(\mathcal{W}_N)$.

Comme on l'a remarqué précédemment, l'énergie est formellement conservée si W est une distribution paire réelle. Rappelons qu'une distribution à valeurs réelles est dite paire si

$$\langle W, \psi \rangle = \langle W, \widetilde{\psi} \rangle, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N; \mathbb{R}),$$

où $\widetilde{\psi}(x) = \psi(-x)$.

Cependant, la conservation de l'énergie n'est pas suffisante pour étudier le comportement en temps long du problème de Cauchy car l'énergie potentielle n'est pas nécessairement positive et la nature non locale du problème nous empêche d'obtenir des bornes ponctuelles. Ce terme peut être contrôlé si on suppose de plus que W est une *distribution positive* ou en supposant que c'est une *distribution définie positive.* Plus précisément, on dit que W est une distribution positive si

$$\langle W, \psi \rangle \geq 0, \quad \forall \psi \geq 0, \; \psi \in C_0^\infty(\mathbb{R}^N; \mathbb{R}),$$

et que c'est une distribution définie positive si

$$\langle W, \psi * \widetilde{\psi} \rangle \geq 0, \quad \psi \in C_0^\infty(\mathbb{R}^N; \mathbb{R}).$$

On vérifiera cette condition en utilisant le fait que pour une distribution paire réelle $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ (voir la proposition 3.2.2)

$$W \text{ est définie positive} \iff \widehat{W} \geq 0 \text{ p.p. sur } \mathbb{R}^N.$$

De plus, on dira qu'une distribution paire réelle $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ est strictement définie positive si

$$\operatorname{inf\,ess} \widehat{W} > 0.$$

Remarque. Ces deux notions sont indépendantes. Comme on le verra après, il existe des distributions positives qui ne sont pas définies positives et il existe des distributions définies positives qui ne sont pas positives.

De (1.14), on remarque que si W est définie positive, on a $E_W(v) \geq 0$ pour tout $v \in \mathcal{E}(\mathbb{R}^N)$. Cependant, dans le cas où W est positive, le terme associé à l'énergie potentielle peut devenir négatif et on n'a donc pas de condition de signe sur $E_W(v)$. Le tableau 1.1 présente un résumé des propriétés vérifiées par les potentiels discutés à la sous-section 1.1.1.

Noyau	Positive	Déf. positive	Strict. déf. positive	$(\mathcal{W}_N)$ est vérifiée
δ	oui	oui	oui	$N \in \{1,2,3\}$
W_ε	oui	oui	non	$N \in \{2,3\}$
$1_{\lvert x\rvert \leq a}$	oui	non	non	$N \geq 1$
$a\delta + bK$	non	oui, si $a \geq \tilde{b} \geq 0$ ou $a \geq -2\tilde{b} \geq 0$	oui, si $a > \tilde{b} \geq 0$ ou $a > -2\tilde{b} \geq 0$	$N = 3$

$\tilde{b} \equiv (4\pi b)/3$

Tableau 1.1 – Propriétés vérifiées par certains noyaux.

Pour établir un résultat sur le caractère bien posé de (GPN), on suit l'approche de [39] et on considère des données initiales u_0 qui appartiennent à l'espace $\phi + H^1(\mathbb{R}^N)$, avec ϕ une fonction d'énergie finie. Plus précisément, on suppose que ϕ est une fonction complexe qui satisfait

$$\phi \in W^{1,\infty}(\mathbb{R}^N), \; \nabla\phi \in H^2(\mathbb{R}^N) \cap C(B^c), \; |\phi|^2 - 1 \in L^2(\mathbb{R}^N), \tag{1.15}$$

où B^c désigne le complémentaire de n'importe quelle boule $B \subseteq \mathbb{R}^N$, de sorte que, en particulier, ϕ satisfait (1.5).

Remarque. On ne suppose pas que ϕ a une limite à l'infini. Dans le cas où $N \in \{1,2\}$ une fonction qui satisfait (1.15) peut avoir des oscillations compliquées, par exemple (voir [41, 40])

$$\phi(x) = \exp(i(\ln(2 + |x|))^{\frac{1}{4}}), \quad x \in \mathbb{R}^2.$$

On note aussi que toute fonction vérifiant (1.15) appartient à l'espace de Sobolev homogène

$$\dot{H}^1(\mathbb{R}^N) = \{\psi \in L^2_{\rm loc}(\mathbb{R}^N) : \nabla\psi \in L^2(\mathbb{R}^N)\}.$$

En particulier, si $N \geq 3$ il existe $z_0 \in \mathbb{C}$ avec $|z_0| = 1$ tel que $\phi - z_0 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ (voir par exemple [59]). En choisissant $\alpha \in \mathbb{R}$ tel que $z_0 = e^{i\alpha}$ et comme l'équation (GPN) est invariant par un changement de phase, on peut supposer que $\phi - 1 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, mais on n'utilise pas explicitement cette décroissance, afin de gérer dans le même temps le cas de la dimension deux.

Notre résultat principal concernant le caractère bien posé du problème de Cauchy est le suivant.

Théorème 8. *Soit W une distribution réelle paire qui vérifie $(\mathcal{W}_N)$.*

(*i*) *On suppose que l'une des conditions suivantes est vérifiée*

(*a*) *$N \geq 2$ et W est une distribution définie positive.*

(*b*) *$N \geq 1$, $W \in \mathcal{M}_{1,1}(\mathbb{R}^N)$ et W est une distribution positive.*

Alors le problème de Cauchy (GPN) *est globalement bien posé dans $\phi + H^1(\mathbb{R}^N)$. Plus précisément, pour tout $w_0 \in H^1(\mathbb{R}^N)$ il existe un unique $w \in C(\mathbb{R}, H^1(\mathbb{R}^N))$, pour lequel $\phi + w$ résout* (GPN) *avec la donnée initiale $u_0 = \phi + w_0$ et pour tout intervalle borné fermé $I \subset \mathbb{R}$, le flot $w_0 \in H^1(\mathbb{R}^N) \mapsto w \in C(I, H^1(\mathbb{R}^N))$ est continu. Par ailleurs, $w \in C^1(\mathbb{R}, H^{-1}(\mathbb{R}^N))$ et l'énergie est conservée*

$$E_W(\phi + w_0) = E_W(\phi + w(t)), \ \forall t \in \mathbb{R}. \tag{1.16}$$

(*ii*) *On suppose que W est strictement définie positive. Alors* (GPN) *est globalement bien posé dans $\phi + H^1(\mathbb{R}^N)$ pour tout $N \geq 1$ et on a la conservation de l'énergie* (1.16).

Par ailleurs, si u est la solution associée à la donnée initiale $u_0 \in \phi + H^1(\mathbb{R}^N)$, on a l'estimation de croissance

$$\|u(t) - \phi\|_{L^2} \leq C|t| + \|u_0 - \phi\|_{L^2},$$

pour tout $t \in \mathbb{R}$, où C est une constante positive qui dépend seulement de E_0, W, ϕ et $\operatorname{inf\,ess} \widehat{W}$.

Finalement on a régularité H^2, c.-à-d. si $u_0 \in \phi + H^2(\mathbb{R}^N)$, alors $u - \phi \in C(\mathbb{R}, H^2(\mathbb{R}^N)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N))$.

On déduit du tableau 1.1 les cas où l'on peut appliquer le théorème 8. Par exemple, en utilisant la partie (ii) du théorème 8, on retrouve les résultats des théorèmes 4 et 5, et aussi l'estimation de croissance prouvée dans [2], dans ce cas-là. D'autre part, la masse de Dirac ne satisfait pas $(\mathcal{W}_N)$ si $N \geq 4$, et donc le théorème 8 ne peut pas être appliqué. En fait, comme il est établi dans les théorèmes 3 et 6, dans ce cas-là une condition de petitesse pour la donnée initiale semble nécessaire.

En outre, si le potentiel converge vers une masse de Dirac, comme c'est le cas des noyaux W_ε donnés par (1.6) lorsque $\varepsilon \to 0$, les solutions correspondantes convergent vers la solution du problème local au sens du résultat suivant.

Proposition 9. *Soient $1 \leq N \leq 3$ et $(W_n)_{n\in\mathbb{N}}$ une suite de distributions paires réelles dans $\mathcal{M}_{2,2}(\mathbb{R}^N) \cap \mathcal{M}_{3,3}(\mathbb{R}^N)$ telles que u_n est la solution globale de* (GPN) *donnée par le théorème 8, avec W_n à la place de W, pour une donnée initiale dans $\phi + H^1(\mathbb{R}^N)$. On suppose que*

$$\lim_{n\to\infty} W_n = W_\infty, \quad \text{dans} \ \ \mathcal{M}_{2,2}(\mathbb{R}^N) \cap \mathcal{M}_{3,3}(\mathbb{R}^N),$$

avec $\|W_\infty\|_{\mathcal{M}_{2,2}\cap\mathcal{M}_{3,3}} > 0$ †. Alors $u_n \to u$ dans $C(I, H^1(\mathbb{R}^N))$, pour tout intervalle fermé borné $I \subset \mathbb{R}$, où u est la solution de (GPN) *avec $W = W_\infty$ et la même donnée initiale.*

Du théorème 8 et de la proposition 7 on tire le résultat suivant pour les noyaux intégrables.

Corollaire 10. *Soit W une fonction réelle paire telle que $W \in L^1(\mathbb{R}^N)$ si $1 \leq N \leq 3$ et $W \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, pour un certain $r > \frac{N}{4}$, si $N \geq 4$. On suppose aussi que l'une des conditions suivantes est vérifiée*

- *$N \geq 1$ et $W \geq 0$ p.p. sur $\mathbb{R}^N$.*
- *$N \geq 2$ et $\widehat{W} \geq 0$ p.p. sur $\mathbb{R}^N$.*

Alors le problème de Cauchy pour (GPN) *est globalement bien posé dans $\phi + H^1(\mathbb{R}^N)$.*

Comme l'a remarqué C. Gallo dans [39], le caractère bien posé dans un espace comme $\phi + H^1(\mathbb{R}^N)$ permet de gérer le problème avec des données initiales dans l'espace d'énergie $\mathcal{E}(\mathbb{R}^N)$ en utilisant des solutions dites *mild*. On rappelle que $u \in C(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ est appelée une solution *mild* de (GPN) si elle vérifie la formule de Duhamel

$$u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}(u(s)(W * (1 - |u(s)|^2)))\, ds, \quad t \in \mathbb{R}.$$

On se reportera à [41, 40] pour des résultats complémentaires sur l'action du semi-groupe de Schrödinger dans $\mathcal{E}(\mathbb{R}^N)$.

Avec les mêmes arguments que [39], on peut également gérer le problème avec des données initiales dans l'espace d'énergie. Par ailleurs, dans le cas $1 \leq N \leq 4$, on prouve que la solution dans l'espace d'énergie avec la condition initiale $u_0 \in \mathcal{E}(\mathbb{R}^N)$, appartient nécessairement à $u_0 + H^1(\mathbb{R}^N)$, sous-ensemble strict de $\mathcal{E}(\mathbb{R}^N)$.

Cela donne aussi l'unicité dans l'espace d'énergie pour $1 \leq N \leq 4$, comme suit.

Théorème 11. *Soit W comme dans le théorème 8. Alors pour tout $u_0 \in \mathcal{E}(\mathbb{R}^N)$, il existe un unique $w \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ tel que $u := u_0 + w$ résout* (GPN)*. Par ailleurs, si $1 \leq N \leq 4$ et $v \in C(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ est une solution* mild *de* (GPN) *avec $v(0) = u_0$, alors $v = u$.*

Enfin, on étudie la conservation du moment et de la masse pour (GPN). Comme cela a été discuté dans plusieurs travaux (voir [7, 9, 77, 10]) les quantités classiques de masse

$$M(u) = \int_{\mathbb{R}^N} (1 - |u|^2)\, dx,$$

et de moment vectoriel

$$P(u) = (P_1(u), \dots, P_N(u)) = \int_{\mathbb{R}^N} \langle\!\langle i\nabla u, u \rangle\!\rangle,$$

†. $(\|\cdot\|_{\mathcal{M}_{2,2}\cap\mathcal{M}_{3,3}} := \max\{\|\cdot\|_{\mathcal{M}_{2,2}}, \|\cdot\|_{\mathcal{M}_{3,3}}\})$

où $\langle\!\langle z_1, z_2 \rangle\!\rangle = \mathrm{Re}(z_1\overline{z}_2)$, sont formellement conservées mais elles ne sont pas bien définies pour tout $u \in \phi + H^1(\mathbb{R}^N)$. Ainsi, il est nécessaire de donner un certain sens généralisé à ces quantités. À la section 3.7, on expliquera en détail une notion de *moment généralisé* et de *masse généralisée* qui permettent d'obtenir les résultats suivants sur les lois de conservation.

Théorème 12. *Soit $N \geq 1$ et $u_0 \in \phi + H^1(\mathbb{R}^N)$. Alors le moment généralisé est conservé par le flot de la solution u de* (GPN) *donnée par le théorème 8.*

Théorème 13. *Soit $1 \leq N \leq 4$. En plus de* (1.15)*, on suppose que $\nabla\phi \in L^{\frac{N}{N-1}}(\mathbb{R}^N)$ si $N \in \{3,4\}$. Si la masse généralisée de $u_0 \in \phi + H^1(\mathbb{R}^N)$ est finie, alors la masse généralisée associée à la solution de* (GPN) *donnée par le théorème 8 est conservée par le flot.*

Un des premiers travaux qui considère un modèle non local pour l'équation de Gross–Pitaevskii est celui de Y. Pomeau et S. Rica [87] qui introduisent le noyau (1.7) pour modéliser un superfluide avec un roton. En fait, la théorie de la superfluidité de l'hélium II de Landau affirme que la courbe de dispersion doit présenter un minimum local non nul appelé roton (voir [71, 37]), ce phénomène a été corroboré plus tard par des observations expérimentales ([34]). Bien que le modèle pris en compte dans [87] a un bon ajustement avec ce minimum, il ne donne pas la vitesse correcte du son. Pour cette raison, N. Berloff dans [5] a proposé le potentiel

$$W(x) = (\alpha + \beta A^2|x|^2 + \gamma A^4|x|^4)\exp(-A^2|x|^2), \quad x \in \mathbb{R}^3, \tag{1.17}$$

où les paramètres A, α, β et γ sont choisis de façon à avoir la bonne vitesse du son. Cependant, l'existence d'un minimum local implique que $\widehat{W}$ doit être strictement négatif dans un voisinage du point où le minimum est atteint. De plus, une simulation numérique dans [5] montre que dans ce cas les solutions présentent un phénomène de concentration de masse qui n'a pas de sens physique. Jusqu'à un certain point, nos résultats sont en accord avec ces observations, dans le sens où le théorème 8 ne peut pas être appliqué au potentiel (1.17), car $\widehat{W}$ et W sont négatifs dans une certaine partie du domaine. Toutefois, on montrera au chapitre 3 que le problème de Cauchy est localement bien posé dans $\phi + H^1(\mathbb{R}^N)$ pour tout noyau pair réel vérifiant $(\mathcal{W}_N)$; en particulier il est localement bien posé pour le potentiel (1.17).

On a démontré que si $\widehat{W} \geq 0$ sur $\mathbb{R}^N$ le problème de Cauchy est globalement bien posé et on a donné des exemples de fonctions positives pour lesquelles sa transformée de Fourier change de signe et les solutions associées restent globales. Par ailleurs, le noyau du paragraphe ci-dessus montre un exemple où le potentiel et sa transformée de Fourier changent de signe, et la solution associée n'est pas globale. C'est une question ouverte d'établir précisément le rôle de ces changements de signe pour l'existence globale des solutions de (GPN).

1.1.3 Les ondes progressives pour l'équation de Gross–Pitaevskii

Une onde progressive qui se propage selon la direction x_1 à une vitesse c est une solution de la forme

$$u_c(x,t) = v(x_1 - ct, x_\perp), \quad x_\perp = (x_2, \dots, x_N).$$

Ainsi on déduit de l'équation (GPN) que le profil de v satisfait

$$ic\partial_1 v + \Delta v + v(W * (1 - |v|^2)) = 0, \quad \text{dans } \mathbb{R}^N. \tag{OPNc}$$

En utilisant la conjugaison complexe, on peut se restreindre au cas $c \geq 0$. On note aussi que toute constante complexe de module un vérifie (OPNc), de sorte qu'on se référera à elles comme les solutions triviales. La question naturelle qu'on se pose pour cette équation est : *quelles sont les conditions sur W et c afin de déterminer s'il existe des solutions d'énergie finie où pas ?*

Dans le cas où W est une masse de Dirac, (OPNc) devient

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0, \quad \text{dans } \mathbb{R}^N, \tag{OPc}$$

et les questions d'existence et non existence ont fait l'objet de nombreux travaux. Les premiers résultats ont été établis dans le programme de C. A. Jones, S. J. Putterman et P. H. Roberts ([62, 61]). En utilisant des développements formels et des simulations numériques, ils ont déterminé qu'en dimension deux et trois, l'équation (OPc) possède des solutions non constantes d'énergie finie, à symétrie axiale autour de l'axe x_1, pour toute vitesse

$$c \in (0, \sqrt{2}),$$

ainsi que pour $c \in \{0\} \cup [\sqrt{2}, \infty)$ les seules solutions d'énergie finie sont les constantes. En utilisant la première composante du moment $p \equiv P_1$, ils tracent les branches des solutions dans le plan énergie E – moment p, en obtenant les figures 1.2 et 1.3. Dans les deux cas ils trouvent une vitesse critique $c^* \in (0, \sqrt{2})$ telle que les solutions possèdent des tourbillons seulement si $c < c^*$.

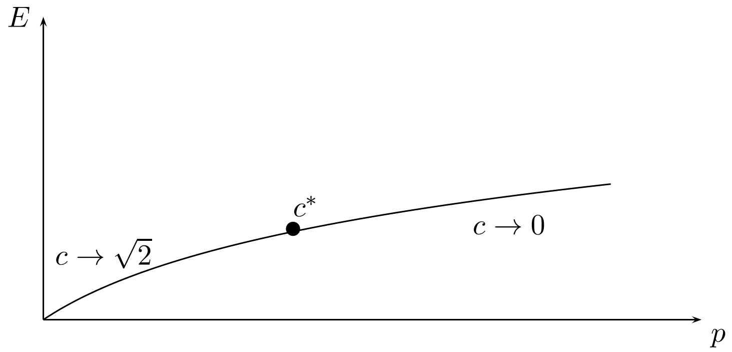

Figure 1.2 – Courbe de l'énergie E en fonction du moment p dans le cas $N = 2$.

Dans le cas de la dimension un, (OPc) est complètement intégrable et les solutions sont explicites.

Théorème 14 ([101, 7]). *Soient $N = 1$, $c \geq 0$ et $v \in \mathcal{E}(\mathbb{R})$ une solution de* (OPc).

(i) Si $c \geq \sqrt{2}$, alors v est une constante de module un.

(ii) Si $0 \leq c < \sqrt{2}$, alors v est soit constante de module un, soit égale, à multiplication par une constante de module un et translation près, à la fonction

$$v_c(x) = \sqrt{1 - \frac{c^2}{2}} \tanh\left(\frac{\sqrt{2 - c^2}}{2}\right) - i\frac{c}{\sqrt{2}}.$$

Dans le cas $N \geq 2$, on a les résultats suivants d'existence et de non existence.

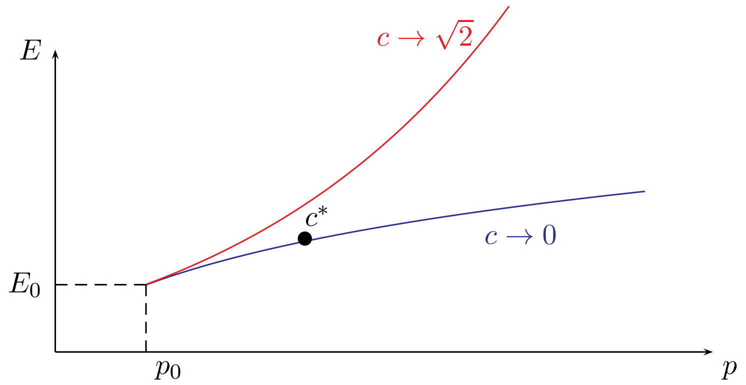

Figure 1.3 – Courbe de l'énergie E en fonction du moment p dans le cas $N = 3$.

Théorème 15 ([22, 12, 47, 49]). *Soit $N \geq 2$ et $v \in \mathcal{E}(\mathbb{R}^N)$ une solution de* (TW*c*)*. On suppose qu'on se place dans l'un des cas suivants*

(i) $c = 0$.

(ii) $c > \sqrt{2}$.

(iii) $N = 2$ *et* $c = \sqrt{2}$.

Alors v est une fonction constante de module un.

Théorème 16 ([12, 11, 25, 8, 77]). *Soit $N \geq 2$. Il existe un ensemble non vide $A \subset (0, \sqrt{2})$ tel que pour tout $c \in A$, il existe une solution non constante de* (TW*c*) *dans $\mathcal{E}(\mathbb{R}^N)$. Par ailleurs, si $N \geq 3$, alors il existe une solution non constante de* (TW*c*) *dans $\mathcal{E}(\mathbb{R}^N)$ pour tout $c \in (0, \sqrt{2})$.*

De cette façon, on voit que $\sqrt{2}$ est une valeur critique pour l'équation (OPc). D'ailleurs, comme $\widehat{\delta} = 1$, elle correspond à la vitesse des ondes sonores donnée par (1.13) :

$$c_s(\delta) = \sqrt{2}.$$

Par rapport au comportement qualitatif des solutions d'énergie finie de (OPc), E. Tarquini a montré dans [99] l'existence d'une valeur minimale $\mathcal{E}(N, c)$ pour l'énergie de Ginzburg–Landau des ondes progressives.

Théorème 17 ([99]). *Soient $N \geq 2$ et $0 < c < \sqrt{2}$. Il existe une constante $\mathcal{E}(N, c) > 0$, qui ne dépend que de la dimension N et de la vitesse c telle que toute solution non constante $v \in \mathcal{E}(\mathbb{R}^N)$ de* (OP*c*) *vérifie*

$$E_{GL}(v) \geq \mathcal{E}(N, c).$$

Par ailleurs,

$$\mathcal{E}(N, c) \to 0, \ \textit{lorsque } c \to \sqrt{2}.$$

En particulier, les seules solutions possibles de (OP*c*) *avec une énergie plus petite que $\mathcal{E}(N, c)$ sont les constantes.*

Bien entendu, le théorème 17 n'empêche pas l'existence d'ondes progressives d'énergie petite. D'ailleurs, dans la figure 1.2 on voit des solutions avec des énergies arbitrairement petites. F.

Béthuel, P. Gravejat et J.-C. Saut [8] ont amélioré ce résultat en dimension trois, en démontrant qu'il existe une énergie minimale indépendante de c. En même temps, ils montrent l'existence de la courbe de solutions en accord avec la figure 1.2 et de la courbe de solutions en bleu dans la figure 1.3.

La preuve de l'énergie minimale dans [8] repose sur l'estimation

$$E_{GL}(v) \geq \left(Kc^2(1+c^2) \int_{\mathbb{R}^N} L_c(\xi)^2 d\xi \right)^{-1}, \text{ pour tout } c \in (0, \sqrt{2}], \tag{1.18}$$

où K est une constante universelle et

$$L_c(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}.$$

L'inégalité (1.18) est valable pour tout $N \geq 2$ et $v \in \mathcal{E}(\mathbb{R}^N)$, solution non constante de (OPc) telle que

$$\inf\{|v(x)| \mid x \in \mathbb{R}^N\} \geq \frac{1}{2}. \tag{1.19}$$

On note que si l'énergie $E(v)$ est petite, l'estimation (1.19) est forcément vérifiée (voir [99, 8]), elle ne représente donc aucune restriction. Un calcul direct montre que

$$\int_{\mathbb{R}^N} L_c(\xi)^2 d\xi = \begin{cases} \dfrac{\pi}{\sqrt{2(2-c^2)}}, & N = 2, \\ \dfrac{\pi^2}{c} \arcsin\left(\dfrac{c}{\sqrt{2}}\right), & N = 3, \\ \infty, & N \geq 4, \end{cases} \tag{1.20}$$

donc (1.18) implique une borne inférieure pour l'énergie seulement en dimension trois. Cependant, il ne donne pas l'existence d'une énergie minimale en dimension supérieure à trois. Le but du chapitre 2 est de montrer qu'en effet il est possible d'étendre ce résultat à toute dimension $N \geq 3$. Plus précisément,

Théorème 18. *Soit $N \geq 3$. Il existe une constante positive $\mathcal{E}(N)$, qui ne dépend que de N, telle que pour toute solution non constante $v \in \mathcal{E}(\mathbb{R}^N)$ de* (OPc)*, on a*

$$E_{GL}(v) \geq \mathcal{E}(N).$$

En particulier, il n'existe pas de solution non constante pour (OPc) *d'énergie petite.*

1.1.4 Non existence pour l'équation (OPNc)

Maintenant on revient à l'équation des ondes progressives non locale (OPNc). Les arguments des sections précédentes suggèrent qu'un résultat de non existence comme celui établi au théorème 15 pourrait être généralisé au cas non local, en utilisant la vitesse des ondes sonores (1.13) comme vitesse critique. Autrement dit, on pourrait conjecturer la non existence de solutions pour des vitesses

$$c > c_s(W) \equiv (2\widehat{W}(0))^{1/2}.$$

Le but du chapitre 4 est de donner une réponse dans cette direction. On travaillera avec des distributions paires réelles définies positives dans $\mathcal{M}_{2,2}(\mathbb{R}^N)$ telles que

(A_1) $\widehat{W}$ est différentiable p.p. sur $\mathbb{R}^N$ et pour tout $j,k \in \{1,\dots,N\}$ l'application

$$\xi \to \xi_j \partial_k \widehat{W}(\xi)$$

est bornée et continue p.p. sur $\mathbb{R}^N$;

(A_2) $\widehat{W}$ est de classe C^2 dans un voisinage de l'origine et $\widehat{W}(0) > 0$.

De plus, si $N \geq 4$, on admet que

$$W \in \mathcal{M}_{N/(N-1),\infty}(\mathbb{R}^N) \cap \mathcal{M}_{2N/(N-2),\infty}(\mathbb{R}^N) \cap \mathcal{M}_{2N/(N-2),2N/(N-2)}(\mathbb{R}^N). \tag{1.21}$$

On remarque que la condition (1.21) est plus restrictive que $(\mathcal{W}_N)$ en dimension $N \geq 4$, mais on en a besoin pour assurer la régularité des ondes progressives. Plus précisément, on prouve au chapitre 4 que si $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$, sous l'hypothèse supplémentaire (1.21) si $N \geq 4$, alors toute solution $v \in \mathcal{E}(\mathbb{R}^N)$ de (OPNc) est régulière et

$$|v(x)| \to 1, \ \nabla v(x) \to 0, \qquad \text{lorsque } |x| \to \infty.$$

D'autre part, la condition (1.21) est vérifiée au moins pour tout $W \in L^1(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$.

On peut maintenant énoncer notre premier résultat de non existence pour l'équation (GPN) de la façon suivante :

Théorème 19. *Soient $N \geq 2$ et $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ une distribution paire réelle définie positive vérifiant (A_1) et (A_2). Si $N \geq 4$, on admet aussi (1.21). On suppose que $c > c_s(W)$ et qu'il existe des constantes $\sigma_1,\dots,\sigma_N \in \mathbb{R}$ telles que*

$$\widehat{W}(\xi) + \alpha_c \sum_{k=2}^{N} \sigma_k \xi_k \partial_k \widehat{W}(\xi) - \sigma_1 \xi_1 \partial_1 \widehat{W}(\xi) \geq 0, \ \textit{pour presque tout } \xi \in \mathbb{R}^N, \tag{1.22}$$

et

$$\sum_{k=2}^{N} \sigma_k + \min\left\{ -\sigma_1 - 1, \frac{\sigma_1 - 1}{\alpha_c + 2}, 2\alpha_c \sigma_j + \sigma_1 - 1 \right\} \geq 0, \tag{1.23}$$

pour tout $j \in \{2,\dots,N\}$, où $\alpha_c := c^2/(c_s(W))^2 - 1$. Alors il n'existe pas de solution non constante pour (OPNc) *dans $\mathcal{E}(\mathbb{R}^N)$.*

Les hypothèses du théorème 19 sont relativement techniques. Cependant, comme nous allons le voir il permet de traiter de nombreux exemples concrets. Pour appliquer le théorème 19 on a besoin de vérifier l'existence des constantes $\sigma_1,\dots,\sigma_N$ vérifiant (1.22) et (1.23). Pour faciliter cette tâche, on donne deux corollaires où les conditions pour la non existence sont exprimées seulement en fonction de W.

Corollaire 20. *Soient $N \geq 2$ et $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ une distribution paire réelle définie positive vérifiant (A_1) et (A_2). Si $N \geq 4$, on admet aussi (1.21). On suppose que $c > c_s(W)$ et que*

$$\widehat{W}(\xi) \geq \max\left\{1, \frac{2}{N-1}\right\} \sum_{k=2}^{N} |\xi_k \partial_k \widehat{W}(\xi)| + |\xi_1 \partial_1 \widehat{W}(\xi)|, \ \textit{pour presque tout } \xi \in \mathbb{R}^N. \tag{1.24}$$

Alors il n'existe pas de solution non constante pour (OPNc) *dans $\mathcal{E}(\mathbb{R}^N)$.*

Corollaire 21. *Soient $N \geq 2$ et $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ une distribution paire réelle définie positive vérifiant (A_1) et (A_2). Si $N \geq 4$, on admet aussi* (1.21)*. On suppose que*

$$c_s(W) < c \leq c_s(W)\left(1 + \inf_{\xi\in\mathbb{R}^N} \frac{(N-1)\widehat{W}(\xi)}{\sum_{k=2}^N |\xi_k \partial_k \widehat{W}(\xi)|}\right)^{1/2}. \tag{1.25}$$

Alors il n'existe pas de solution non constante pour (OPNc) *dans* $\mathcal{E}(\mathbb{R}^N)$.

Concernant les ondes statiques, on a le résultat suivant.

Théorème 22. *Soient $N \geq 2$ et $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ une distribution paire réelle définie positive vérifiant (A_1). Si $N \geq 4$, on admet aussi* (1.21)*. On suppose que $c = 0$ et que*

$$\xi_j \partial_j \widehat{W}(\xi) \leq 0, \text{ pour presque tout } \xi \in \mathbb{R}^N, \tag{1.26}$$

pour tout $j \in \{1, \dots, N\}$. Alors il n'existe pas de solution non constante pour (OPNc) *dans* $\mathcal{E}(\mathbb{R}^N)$.

Si l'on considère le potentiel $W = a\delta$, $a > 0$, on a que $\widehat{W} = a$ et donc $\nabla\widehat{W} = 0$. De cette façon les conditions (1.24), (1.25) et (1.26) sont vérifiées et en conséquence on peut appliquer le corollaire 20 ou 21 et le théorème 22 pour conclure la non existence de solutions non triviales pour l'équation (OPNc) pour tout

$$c \in \{0\} \cup (\sqrt{2a}, \infty). \tag{1.27}$$

En particulier, en prenant $a = 1$, on retrouve le théorème 15 dans les cas (i) et (ii).

Considère maintenant une perturbation de la masse de Dirac en dimension $N \in \{2, 3\}$:

$$W_\varepsilon = \delta + \varepsilon f, \quad \varepsilon \geq 0,$$

où f est une fonction paire réelle telle que $f, |x|^2 f, |x||\nabla f| \in L^1(\mathbb{R}^N)$, de sorte que $\widehat{W}_\varepsilon = 1 + \varepsilon\widehat{f} \in C^2(\mathbb{R}^N)$. Puisque

$$\widehat{x_j \partial_k f} = -(\delta_{j,k}\widehat{f} + \xi_k \partial_j \widehat{f}),$$

on a

$$\|\widehat{f}\|_{L^\infty(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)}, \quad \|\xi_k \partial_j \widehat{f}\|_{L^\infty(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)} + \|x_j \partial_k f\|_{L^1(\mathbb{R}^N)}.$$

Ainsi W satisfait les conditions (A_1) et (A_2) si $\varepsilon < \|f\|_{L^1(\mathbb{R}^N)}^{-1}$, et dans ce cas la vitesse des ondes sonores

$$c_s := c_s(W) = \left(2 + 2\varepsilon \int_{\mathbb{R}^N} f\right)^{1/2},$$

est bien définie. En outre, (1.24) est vérifiée si

$$\varepsilon < \left(4\|f\|_{L^1(\mathbb{R}^N)} + \sum_{k=1}^N \|x_k \partial_k f\|_{L^1(\mathbb{R}^N)}\right)^{-1}. \tag{1.28}$$

Par conséquent, sous la condition (1.28), le corollaire 20 implique la non existence de solutions non triviales de (OPNc) dans $\mathcal{E}(\mathbb{R}^N)$ pour tout $c \in (c_s, \infty)$.

Comme autre exemple, on reprend le noyau (1.6), dont la transformée de Fourier est

$$\widehat{W}_\varepsilon(\xi) = \frac{1}{(1+\varepsilon^2|\xi|^2)}.$$

De façon plus générale, on considère les noyaux $W_{a,b}$ de la forme

$$\widehat{W}_{a,b}(\xi) = \rho_{a,b}(r) \equiv \frac{1}{(1+ar^2)^{b/2}}, \qquad r = |\xi|,\ a,b > 0,$$

donc

$$c_s := c_s(W_{a,b}) = \sqrt{2}.$$

De cette façon, $W_{a,b}$ est une distribution paire réelle définie positive et puisque $\widehat{W}_{a,b} \in L^\infty(\mathbb{R}^N)$, $W_{a,b} \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ pour $N \in \{2,3\}$. De plus, l'hypothèse (A_2) est satisfaite. De plus, on peut vérifier que $W_{a,b} \in L^1(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$ pour tout $N \geq 4$ et $b > N-1$, ce qui implique (1.21). D'autre part, un calcul élémentaire montre que (A_1) et (1.26) sont vérifiés pour tout $a, b > 0$. Le théorème 22 implique donc la non existence d'ondes statiques ($c = 0$) non triviales de (OPNc) dans $\mathcal{E}(\mathbb{R}^N)$ dans les cas suivants

(a) $N = 2$ ou 3 ;

(b) $N \geq 4$, $b > N - 1$.

De plus, en utilisant les corollaires 20 et 21 on peut calculer les intervalles suivants pour la vitesse où il n'existe pas de solution non constante de (OPNc) dans $\mathcal{E}(\mathbb{R}^N)$:

(c) $N = 2$, $b \leq 1/2$, $c \in (c_s, \infty)$;

(d) $N = 2$, $b > 1/2$, $c \in (c_s, \sqrt{2+2/b}\,]$;

(e) $N = 3$, $b \leq 1$, $c \in (c_s, \infty)$;

(f) $N = 3$, $b > 1$, $c \in (c_s, \sqrt{2+2/b}\,]$;

(g) $N \geq 4$, $b > N-1$, $c \in (c_s, \sqrt{2+2/b}\,]$.

On remarque que si $b \to 0$, $\widehat{W}_{a,b} \to 1$ et donc $W \to \delta$, au sens des distributions. Ainsi les cas (a), (c) et (e) peuvent être vus comme une généralisation du théorème 15 dans les cas (i) et (ii). Lorsque b augmente, on a une borne supérieure pour l'intervalle de la vitesse. Une interprétation possible de cette borne est que lorsque la valeur de b augmente, le noyau devient de plus en plus épars dans le sens où sa masse est de moins en moins concentrée proche de l'origine et l'effet non local est donc moins localisé. Cependant, nous ne savons pas si ce type de borne supérieure est simplement une conséquence de notre preuve ou si en fait, pour des potentiels particuliers, il pourrait même exister plusieurs intervalles de vitesses où on a existence et non existence.

Dans le théorème 19 on a admis que $\widehat{W}$ est de classe C^2 dans un voisinage de l'origine. Cependant, on voit que le noyau (1.8) ne satisfait pas cette condition. En fait, il n'est même pas continu en zéro et la vitesse des ondes sonores n'est pas correctement définie. Une question naturelle est de savoir s'il est possible d'affaiblir cette hypothèse sur la régularité. Le point clé où apparaît que $\widehat{W}$ est de classe C^2 est l'étude des ensembles

$$\Gamma_{j,c} \equiv \{\nu = (\nu_1,\nu_2) \in \mathbb{R}^2 : |\nu|^4 + 2\widehat{W}(\nu_1 e_1 + \nu_2 e_j)|\nu|^2 - c^2\nu_1^2 = 0\}, \quad j \in \{2,\dots,N\},$$

où $\{e_k\}_{k\in\{1,\dots,N\}}$ est la base canonique de $\mathbb{R}^N$, qui jouent un rôle fondamental dans la preuve du théorème 19. De façon générale, si $\widehat{W}$ est de classe C^2, on peut appliquer le lemme de Morse et en déduire que si $c > c_s(W)$, alors, dans un voisinage de l'origine, $\Gamma_{j,c}$ est décrit par deux courbes $\gamma^+_{j,c}$ et $\gamma^+_{j,c}$, dont l'allure est donnée dans la figure 1.4 et vérifiant

$$\ell_{j,c} \equiv \lim_{t\to 0^+} \left(\frac{\gamma^+_{j,c}(t)}{t}\right)^2 = \lim_{t\to 0^+} \left(\frac{\gamma^-_{j,c}(t)}{t}\right)^2.$$

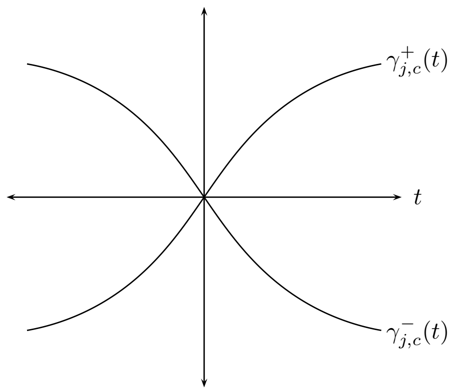

Figure 1.4 – L'ensemble $\Gamma_{j,c}$ au voisinage de l'origine pour $\widehat{W}$ de classe C^2.

En conclusion, si W n'est pas de classe C^2 mais l'ensemble $\Gamma_{j,c}$ satisfait les propriétés mentionnées ci-dessus, on peut établir une généralisation du théorème 19. On précise cette discussion en introduisant la condition :

(A_3) Pour tout $j \in \{2, \dots, N\}$ et $c > 0$, il existe $\delta > 0$ et deux fonctions $\gamma^+_{j,c}$ et $\gamma^-_{j,c}$, définies sur l'intervalle $(0, \delta)$, tels que l'ensemble $\Gamma_{j,c} \cap B(0, \delta)$ est de mesure de Lebesgue nulle, $\gamma^\pm_{j,c} \in C^1((0, \delta))$, et

$$\gamma^+_{j,c}(t) > 0, \quad \gamma^-_{j,c}(t) < 0, \quad (t, \gamma^\pm_{j,c}(t)) \in \Gamma_{j,c}, \quad \text{pour tout } t \in (0, \delta).$$

Par ailleurs, les limites suivantes existent et sont égales

$$\lim_{t\to 0^+} \left(\frac{\gamma^+_{j,c}(t)}{t}\right)^2 = \lim_{t\to 0^+} \left(\frac{\gamma^-_{j,c}(t)}{t}\right)^2 =: \ell_{j,c}.$$

On peut maintenant établir une version générale du théorème 19.

Théorème 23. *Soient $N \geq 2$, $c > 0$ et $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ une distribution paire réelle définie positive vérifiant (A_1) et (A_3) avec*

$$\ell_c := \ell_{1,c} = \ell_{2,c} = \cdots = \ell_{N,c} > 0. \tag{1.29}$$

Si $N \geq 4$, on admet aussi (1.21). On suppose qu'il existe des constantes $\sigma_1, \dots, \sigma_N \in \mathbb{R}$ telles que

$$\widehat{W}(\xi) + \ell_c \sum_{k=2}^{N} \sigma_k \xi_k \partial_k \widehat{W}(\xi) - \sigma_1 \xi_1 \partial_1 \widehat{W}(\xi) \geq 0, \ \textit{pour presque tout } \xi \in \mathbb{R}^N, \tag{1.30}$$

et

$$\sum_{k=2}^{N} \sigma_k + \min\left\{-\sigma_1 - 1, \frac{\sigma_1 - 1}{\ell_c + 2}, 2\ell_c\sigma_j + \sigma_1 - 1\right\} \geq 0, \tag{1.31}$$

pour tout $j \in \{2, \ldots, N\}$*. Alors il n'existe pas de solution non constante pour* (OPNc) *dans* $\mathcal{E}(\mathbb{R}^N)$.

On remarque que la condition (1.29) est nécessaire pour que les solutions soient non constantes (voir chapitre 4).

On rappelle que le potentiel (1.8)–(1.9) est donné par

$$W = a\delta + bK, \quad \text{avec} \quad K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}, \quad x \in \mathbb{R}^3 \backslash \{0\}.$$

Par le tableau 1.1, $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ est une distribution paire réelle définie positive pour tous $a, \tilde{b}$ tels que

$$a \geq \tilde{b} \geq 0 \qquad \text{ou} \qquad a > -2\tilde{b} \geq 0, \tag{1.32}$$

où $\tilde{b} = (4\pi b)/3$. De plus, un calcul élémentaire montre que (A_1) est aussi vérifié. On explique maintenant comment le théorème 23 permet de déduire la non existence de solutions non triviales pour (OPNc) dans $\mathcal{E}(\mathbb{R}^N)$, pour tout

$$(2\max\{a - \tilde{b}, a\})^{1/2} < c < \infty, \tag{1.33}$$

à condition que $a > 0$ et que l'une des deux conditions apparaissant dans (1.32) soit vérifiée.

En fait, puisque la transformée de Fourier de W est

$$\widehat{W}(\xi) = a + \tilde{b}\left(\frac{3\xi_3^2}{|\xi|^2} - 1\right), \qquad \xi \in \mathbb{R}^3 \backslash \{0\},$$

on voit que la courbe $\Gamma_{2,c}$ est donnée par une fonction régulière et on peut donc se restreindre au cas où le lemme de Morse s'applique, ce qui fournit l'existence des courbes $\gamma_{2,c}^{\pm}$ avec $\ell_{2,c} = c^2/(2a) - 1$. Par contre, $\Gamma_{3,c}$ est décrit par l'équation

$$(x^2 + y^2)^2 + 2\widehat{W}(xe_1 + ye_3)(x^2 + y^2) - c^2x^2 = 0. \tag{1.34}$$

Cependant, (1.34) est une équation algébrique que l'on peut résoudre explicitement et conclure que les courbes $\gamma_{3,c}^{\pm}$ sont données par

$$\gamma_{3,c}^{\pm}(t) = \pm\sqrt{-t^2 - a - 2\tilde{b} + \sqrt{6\tilde{b}t^2 + (a + 2\tilde{b})^2 + c^2t^2}},$$

pour tout $|t| < c^2 - 2(a - \tilde{b})$. Par conséquent (A_3) est satisfait et

$$\ell_{3,c} = -1 + (6\tilde{b} + c^2)/(2(a + 2\tilde{b})).$$

On note que par (1.32), $\ell_{3,c}$ est une constante positive bien définie et que la condition (1.29), i.e. $\ell_{3,c} = \ell_{2,c}$, implique que

$$(c^2 - 3a)b = 0.$$

Le cas $b = 0$ a déjà été traité (voir (1.27)). Si $b \neq 0$, on a que $c^2 = 3a$ et donc $\ell_c := \ell_{2,c} = \ell_{3,c} = 1/2$. Ensuite, en prenant $\sigma_1 = 0$ et $\sigma_2 = \sigma_3 = 1/2$, (1.31) est satisfait et le membre à droite de l'inégalité (1.30) devient

$$a + \tilde{b}\left(3\frac{\xi_3^2}{|\xi|^2}\left(1 - \frac{\xi_2^2}{2|\xi|^2}\right) - 1\right) + \frac{3\tilde{b}}{2}\frac{\xi_3^2}{|\xi|^2}\left(1 - \frac{\xi_3^2}{|\xi^2|}\right),$$

qui est positif par (1.32). Par conséquent, on conclut du théorème 23 qu'il n'existe pas de solution non triviale de (OPNc) dans $\mathcal{E}(\mathbb{R}^N)$, pour $a > 0$ et b vérifiant (1.32) et (1.33).

1.2 L'équation de Landau–Lifshitz

L'équation de Landau–Lifshitz a été introduite par L. Landau et E. Lifshitz dans [72] pour décrire la dynamique de l'aimantation dans un milieu ferromagnétique. Elle s'exprime sous la forme

$$\partial_t m + m \times F_{\text{eff}}(m) = 0, \quad m(t,x) \in \mathbb{S}^2,\ t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{1.35}$$

où $m = (m_1, m_2, m_3)$ représente le vecteur de densité d'aimantation et F_{eff} est le champ effectif, c.-à-d. moins le gradient L^2 de l'énergie

$$E_{LL}(m) = E_e(m) + E_{\text{ani}}(m).$$

Notons qu'une des particularités essentielles du modèle est la contrainte $m(t,x) \in \mathbb{S}^2$, qui exprime que la norme du vecteur m reste constante et unitaire ($|m(x,t)| = 1$). Les deux termes qui comprennent l'énergie totale E_{LL} sont l'énergie d'échange :

$$E_e(m) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla m|^2\, dx,$$

et l'énergie d'anisotropie :

$$E_{\text{ani}}(m) = \frac{1}{2}\int_{\mathbb{R}^N} \tilde{e}(m)\, dx.$$

Cette dernière dépend du type de milieu qu'on considère ([60, 66]). Les modèles les plus fréquents sont

$$\tilde{e}(m) = 0, \qquad \text{(milieu isotrope)}, \tag{1.36}$$

$$\tilde{e}(m) = 1 - m_3^2, \qquad \text{(milieu anisotrope uniaxial dans la direction } e_3\text{)}, \tag{1.37}$$

$$\tilde{e}(m) = m_3^2, \qquad \text{(milieu anisotrope planaire)}. \tag{1.38}$$

En termes de l'équation (1.35), ces densités donnent

$$\partial_t m + m \times (\Delta m + \lambda m_3 e_3) = 0, \quad m(t,x) \in \mathbb{S}^2,\ t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{LL$_\lambda$}$$

où $\lambda = 0$, $\lambda = 1$ et $\lambda = -1$ correspondent respectivement aux cas (1.36), (1.37), (1.38). De plus, si m est régulier, en différenciant deux fois la condition $|m(t,x)|^2 = 1$, on vérifie que $m \cdot \Delta m = -|\nabla m|^2$, de sorte qu'en prenant le produit vectoriel de m avec l'équation (1.35), on peut réécrire (1.35) sous la forme

$$m \times \partial_t m = \Delta m + |\nabla m|^2 m + \lambda(m_3 e_3 - m_3^2 m). \tag{1.39}$$

Ainsi, dans les cas d'un matériel ferromagnétique isotrope ($\lambda = 0$), on retrouve l'équation des *Schrödinger maps*. Ce type d'équations a été intensivement étudiée en raison de ses applications dans plusieurs domaines de la physique et des mathématiques. On se reportera à [53] pour un aperçu et à [3] pour quelques résultats récents sur le caractère bien posé du problème de Cauchy.

En matière de solutions particulières, plusieurs travaux de physiciens montrent l'existence de solutions localisées intéressantes, surtout en dimension deux, où ils trouvent des solutions de vorticité non triviale. Cependant, il n'y a pas beaucoup de résultats rigoureux d'un point de vue mathématique. Nous rappelons que pour $N = 2$, la charge magnétique est définie par

$$w(v) = \langle v, \partial_1 v \times \partial_2 v \rangle.$$

Ainsi, si v est une fonction d'énergie finie, constante à l'infinie, son degré $\mathbb{S}^2$ est donné par

$$d(v) = \frac{1}{4\pi} \int_{\mathbb{R}^2} w(v)\, dx.$$

Cette quantité prend des valeurs entières qui coïncident avec le degré topologique de l'application $v \circ \Pi : \mathbb{S}^2 \to \mathbb{S}^2$, où Π se réfère à la projection stéréographique par rapport au pôle nord $(0, 0, 1)$ (voir [19]). De plus, on a la borne inférieure suivante

$$\mathcal{E}_e(v) \geq 4\pi |d(v)|, \tag{1.40}$$

pour tout $v \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{S}^2) \cap \dot{H}^1(\mathbb{R}^2)$. Dans le cas isotrope $\lambda = 0$, un exemple de solutions d'énergie finie sont les *instantons* de Belavin–Polokov $Q^n = (Q_1^n, Q_2^n, Q_3^n)$ donnés par

$$Q_1^n + iQ_2^n = (x_1 + ix_2)^n, \; Q_3^n = \frac{1 - (x_1^2 + x_2^2)^n}{1 + (x_1^2 + x_2^2)^n}, \quad n \in \mathbb{Z}, \tag{1.41}$$

de degré $d(Q^n) = n$, pour lesquels l'inégalité (1.40) est atteinte, c.-à-d.

$$\mathcal{E}_e(Q^n) = 4\pi |n|.$$

On référera à [4, 89] pour les détails de ces calculs et à [70] pour d'autres exemples de solutions statiques explicites dans le cas isotrope.

Lorsque le matériel présente une anisotropie ($\lambda \neq 0$), un argument formel du type Derrick–Pohozaev ([33, 89, 84]) montre que l'existence d'une solution statique v (indépendante de t) en dimension $N \geq 2$ implique que

$$(N-2)E_e(v) + \lambda N E_a(v) = 0.$$

Cette relation suggère que dans le cas $N = 2$, l'anisotropie empêche l'existence de solutions statiques d'énergie finie. En utilisant des méthodes numériques, B. Piette et W. Zakrzewski ([85]) trouvent des ondes solitaires périodiques en temps de degré $n \in \mathbb{Z}$ pour l'équation (1.35) dans le cas $N = 2$ et $\lambda = 1$. Plus précisément, en coordonnées sphériques

$$(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)), \theta \in [0, \pi], \; \phi \in [0, 2\pi), \tag{1.42}$$

elles s'expriment sous la forme

$$\theta = \theta_w(r), \; \phi = n\chi - \omega t + \phi_0, \tag{1.43}$$

où r et χ sont les coordonnées polaires dans $\mathbb{R}^2$, et ϕ_0 une fonction quelconque en raison de l'invariance de l'équation par un changement de phase. Plus tard, ce résultat a été rigoureusement prouvé par S. Gustafson et J. Shatah [55]. D'autre part, X. Pu et B. Guo [88] ont montré que la présence de l'anisotropie est fondamentale, au sens où si $\lambda = 0$, il n'existe pas de solution d'énergie finie de l'équation (1.35) pour $N = 2$ sous la forme (1.43).

Dans la présence d'anisotropie planaire ($\lambda = -1$), l'existence de solutions localisées a été analysée par N. Papanicolaou et P. N. Spathis dans [83], dans les cas $N = 2$ et $N = 3$, en utilisant des développements formels et des méthodes numériques. Ils cherchaient des ondes progressives d'énergie finie qui se propagent à vitesse c selon l'axe x_1, c.-à-d. qui s'expriment sous la forme

$$m_c(x,t) = u(x_1 - ct, x_2, \dots, x_N),$$

de façon qu'elles résolvent

$$-\Delta u = |\nabla u|^2 u + u_3^2 u - u_3 e_3 + cu \times \partial_1 u. \tag{OPLL$_c$}$$

Notons que le changement de variable $m \mapsto -m$ permet de se réduire au cas $c \geq 0$. Dans [83] ils montrent qu'il existe une branche de solutions de (OPLL$_c$) à symétrie axiale autour de l'axe x_1, pour toute vitesse $c \in (0,1)$ et ils conjecturent aussi qu'il n'y a pas de solution non triviale pour c supérieur ou égal à 1. Même si ces solutions sont de degré nul ($d(u) = 0$), elles ont des secteurs à topologie non triviale. Plus précisément, dans le cas de la dimension deux, dans [83] les auteurs calculent une vitesse critique $c^* \approx 0.78$ telle que pour $c < c^*$, il existe exactement deux points qu'on appellera tourbillons (ou *vortices*) $q^\pm = (\pm a_c, 0)$, ($a_c > 0$) tels que $u_3(q^\pm) = 1$ et $|u_3| < 1$ sur $\mathbb{R}^2 \setminus \{q^\pm\}$. De plus, loin de ces deux points, u_3 est presque nulle et la fonction u recouvre l'hémisphère supérieur de la sphère $\mathbb{S}^2$. D'ailleurs, si on considère la fonction (à valeurs complexes)

$$\psi = \frac{u_1 + iu_2}{1 + u_3},$$

on vérifie que

$$\Delta\psi + \frac{1 - |\psi|^2}{1 + |\psi|^2}\psi + ic\partial_1\psi = \frac{2\bar{\psi}}{1 + |\psi|^2}(\nabla\psi \cdot \nabla\psi). \tag{1.44}$$

Cette équation ressemble fortement à l'équation des ondes progressives de Gross–Pitaevskii (OPNc). D'autre part, la fonction ψ s'annule sur $q^\pm$ et autour de chaque point $q^\pm$ le degré $\mathbb{S}^1$ de $\psi/|\psi|$ est

$$\deg\left(\frac{\psi}{|\psi|}, \partial B(q^\pm, r), \mathbb{S}^1\right) = \frac{1}{2\pi}\int_{\partial B(q^\pm, r)} \partial_\tau \phi^\pm = \pm 1,$$

pour $r > 0$ petit, où $\psi = |\psi| e^{i\phi^\pm}$ sur $\partial B(q^\pm, r)$ et ∂_τ est la dérivée tangentielle. On conclut qu'aux points $q^\pm$ la fonction ψ a deux tourbillons, de degré 1 et -1 chacun.

Pour des vitesses $c \in (c^*, 1)$, on a que $\|u_3\|_{L^\infty(\mathbb{R}^2)} < 1$ et il n'y a donc pas de tourbillons. En utilisant la courbe de dispersion de l'énergie en fonction de la première composante du moment vectoriel

$$p(u) = -\int_{\mathbb{R}^2} x_2 w(u),$$

ces solutions sont représentées dans la figure 1.5. En particulier, on voit que la courbe présente un minimum non nul.

Le cas de la dimension trois est similaire avec une valeur critique $c^* \approx 0,93$ telle que les solutions de (OPLL$_c$) présentent cette fois-ci une structure tourbillonnaire en forme d'anneau

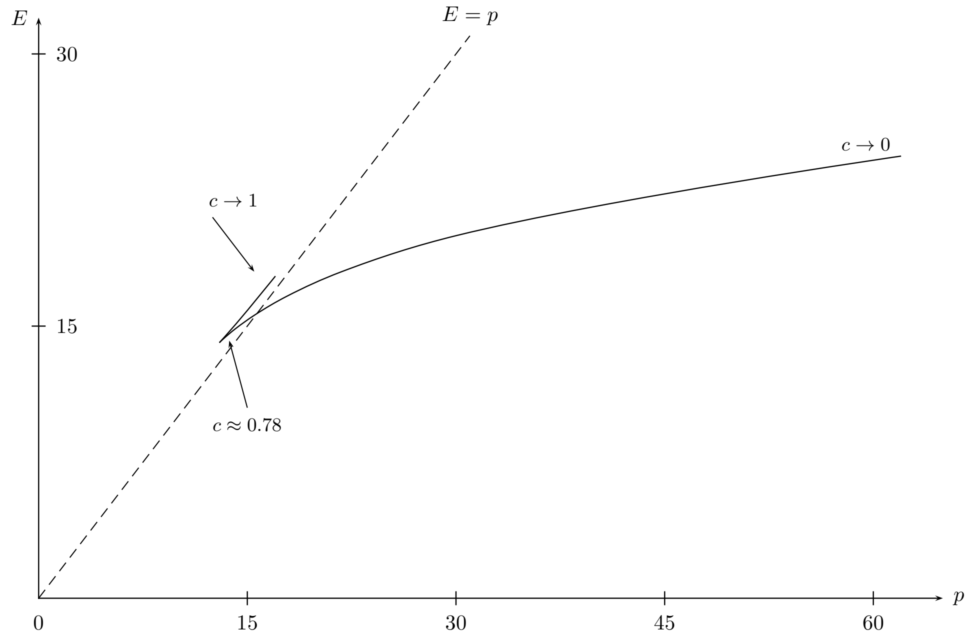

Figure 1.5 – Courbe de l'énergie E en fonction du moment p dans le cas de la dimension deux.

(*vortex ring*) seulement pour $c < c^*$. Le dessin de la courbe énergie-moment en dimension trois ressemble aussi à la figure 1.5.

En comparant ces types de solutions avec les résultats décrits pour les ondes progressives à la sous-section 1.1.3 pour l'équation de Gross–Pitaevskii, on reconnait le même type de phénomènes et les travaux de N. Papanicolaou et P. N. Spathis peuvent être considérés comme une généralisation des études de C. A. Jones, S. J. Putterman et P. H. Roberts dans [62, 61]. Cependant, il est important de remarquer une différence fondamentale : la courbe de dispersion énergie–moment pour l'équation de Gross–Pitaevskii tend vers zéro lorsque $p \to 0$ dans les cas de la dimension deux (voir la figure 1.2). Un des objectifs principaux du chapitre 5 est précisément d'établir que la courbe des solutions reste éloignée de l'origine (voir figure 1.5).

Récemment, F. Lin et J. Wei [74] ont prouvé rigoureusement l'existence de solutions d'énergie finie de l'équation (OPLL$_c$), pour c petit en dimension deux et trois, par des arguments perturbatifs. Une autre approche pour montrer l'existence pourrait être de considérer le problème de minimisation de l'énergie sous la contrainte du moment fixé, comme on en discutera à la sous-section suivante.

1.3 Étude des ondes progressives de l'équation de Landau–Lifshitz planaire

Dans le reste de cette introduction nous considérons le problème des ondes progressives pour l'équation de Landau–Lifshitz à anisotropie planaire, c.-à-d. les solutions de l'équation (OPLL$_c$).

On note que toute constante dans $\mathbb{S}^1 \times \{0\}$ vérifie (OPLL$_c$), nous les appellerons les solutions triviales de (OPLL$_c$). De plus, on s'intéresse à des solutions d'énergie finie

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} v_3^2 < +\infty,$$

donc l'espace naturel associé est

$$\mathcal{E}(\mathbb{R}^N) = \{v \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^3) : \nabla v \in L^2(\mathbb{R}^N),\ v_3 \in L^2(\mathbb{R}^N),\ |v| = 1 \text{ p.p. sur } \mathbb{R}^N\}.$$

On remarque que dans le cas $N = 1$, l'équation (OPLL$_c$) est complètement intégrable et que l'on peut calculer les solutions d'énergie finie explicitement de la forme suivante :

Proposition 24 ([79, 81, 91])**.** *Soient $N = 1$, $c \geq 0$ et $u \in \mathcal{E}(\mathbb{R})$ une solution de* (OPLL$_c$).

(*i*) *Si $c \geq 1$, alors u est constante.*

(*ii*) *Si $0 \leq c < 1$, alors u est soit constante dans $\mathbb{S}^1 \times \{0\}$, soit égale (à translation de u et à rotation complexe de $\check{u} \equiv u_1 + iu_2$ près) à la fonction*

$$u_1 = c\,\mathrm{sech}(\sqrt{1-c^2}\,x), \quad u_2 = \tanh(\sqrt{1-c^2}\,x), \quad u_3 = \pm\sqrt{1-c^2}\,\mathrm{sech}(\sqrt{1-c^2}\,x).$$

De plus, si $0 < c < 1$, on a les relations suivantes entre l'énergie E, le moment p et la vitesse c :

$$E(u) = 2\sqrt{1-c^2}, \quad E(p) = 2\sin\left(\frac{p}{2}\right) \quad \text{et} \quad \frac{dE}{dp} = \cos(p) = c.$$

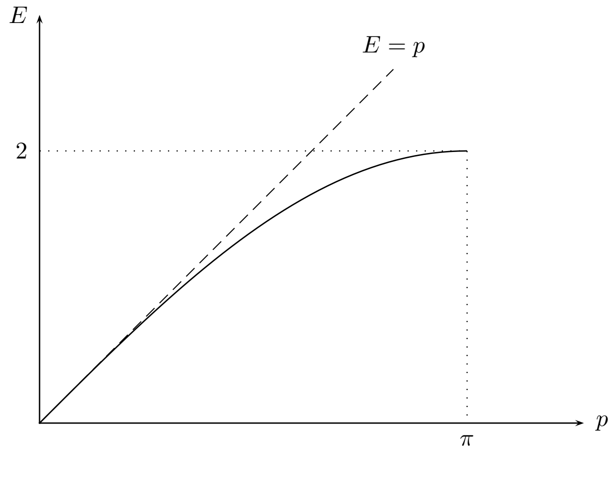

Figure 1.6 – Courbe de l'énergie E en fonction du moment p dans le cas de la dimension un.

La figure 1.6 montre l'énergie comme fonction du moment. En particulier, on note qu'il existe des solutions d'énergie petite, mais qu'il y a une valeur maximale pour l'énergie et pour le moment, afin d'avoir des solutions non triviales.

1.3.1 Le moment et la courbe minimisante en dimension deux

Le moment vectoriel (en dimension deux) $P = (P_1, P_2)$ est donné par

$$P_1(v) = -\int_{\mathbb{R}^2} x_2 w(v), \quad P_2 = \int_{\mathbb{R}^2} x_1 w(v), \tag{1.45}$$

qui, au moins formellement, est conservé par le flot des solutions de l'équation (1.35). Cependant P n'est pas bien défini dans $\mathcal{E}(\mathbb{R}^2)$, puisque l'application

$$v \in \mathcal{E}(\mathbb{R}^2) \to x_2 w(v) \in \mathbb{R}$$

n'est pas nécessairement intégrable. Par contre, si on suppose qu'il existe $R \equiv R(u)$ tel que l'on a le relèvement

$$\check{v} \equiv v_1 + iv_2 = \varrho e^{i\theta}, \text{ sur } B(0,R)^c, \tag{1.46}$$

où $\varrho \equiv \sqrt{v_1^2 + v_2^2} = \sqrt{1 - v_3^2}$ et $\varrho, \theta \in \dot{H}^1(B(0,R)^c)$, il en découle que la vorticité est donnée par

$$w(v) = -\operatorname{rot}(v_3 \nabla \theta),$$

où $\operatorname{rot}(f_1, f_2) = \partial_1 f_2 - \partial_2 f_1$. Si on admet que le relèvement est valable sur tout $\mathbb{R}^2$, en intégrant formellement par parties, on déduit que

$$P_1(v) = \int_{\mathbb{R}^2} v_3 \partial_1 \theta, \quad P_2(v) = \int_{\mathbb{R}^2} v_3 \partial_1 \theta. \tag{1.47}$$

D'autre part, pour $j \in \{1, 2\}$,

$$|v_3 \partial_j \theta| \leq \frac{|v_3||1 - v_3^2|^{\frac{1}{2}} |\partial_j \theta|}{(1 - \|v_3\|^2_{L^\infty(\mathbb{R}^2)})^{1/2}} \leq \frac{e(v)}{(1 - \|v_3\|^2_{L^\infty(\mathbb{R}^2)})^{1/2}},$$

où $e(v)$ est la densité d'énergie

$$e(v) \equiv \frac{1}{2}(|\nabla v|^2 + v_3^2) = \frac{1}{2}\Big(\frac{|\nabla v_3|^2}{1 - v_3^2} + \big(1 - v_3^2\big)|\nabla \theta|^2 + v_3^2\Big).$$

On en déduit alors que l'expression pour le moment donné par (1.47) est bien définie lorsque la fonction a un relèvement global. Cependant l'existence de ce relèvement est une question de nature topologique. Cette question est en fait liée au degré $\mathbb{S}^1$ de l'application

$$\frac{\check{v}}{|\check{v}|} : \partial B(0,R) \to \mathbb{S}^1,$$

lorsque $\check{v}/|\check{v}|$ est bien défini. On remarque que ces types de problèmes ressemblent à ceux que l'on a déjà rencontrés dans l'étude des ondes progressives pour l'équation de Gross–Pitaevskii (voir [7, 77, 10, 31]). Pour pallier à cet inconvénient, nous considérons l'espace

$$\tilde{\mathcal{E}}(\mathbb{R}^2) = \{v \in \mathcal{E}(\mathbb{R}^2) : \exists R \geq 0 \text{ t.q. } \|v_3\|_{L^\infty(B(0,R))^c)} < 1\}.$$

et comme on le montrera au chapitre 5, pour tout $v \in \tilde{\mathcal{E}}(\mathbb{R}^2)$, il existe $R \equiv R(v)$ tel que l'on a le relèvement (1.46). De plus, on peut donner une notion de *moment généralisé* valable pour tout $v \in \tilde{\mathcal{E}}(\mathbb{R}^2)$. D'autre part, on verra que toute solution de (OPLL_c) dans $\mathcal{E}(\mathbb{R}^2)$ appartient aussi à $\tilde{\mathcal{E}}(\mathbb{R}^2)$, cette définition sera alors suffisamment générale pour énoncer nos résultats.

Une difficulté supplémentaire de la définition du moment est sa non invariance par translation. En fait, en utilisant la fonction de translation τ_a définie par

$$\tau_a f(\cdot) = f(\cdot - a), \quad a = (a_1, a_2) \in \mathbb{R}^2,$$

on a pour $p \equiv P_1$

$$p(\tau_a u) = p(u) - 4\pi a_2 d(u).$$

Cependant, les ondes progressives trouvées par N. Papanicolaou et P. N. Spathis dans [83] sont de degré zéro ($d(u) = 0$), donc on pourrait se restreindre à ce type de solutions.

En outre, au moins formellement, une méthode possible pour construire une solution $u \in \mathcal{E}(\mathbb{R}^2)$ pour (OPLL$_c$), avec un moment prescrit $p(u) \equiv P_1(u) = \mathfrak{p}$, est de considérer le problème de minimisation

$$\inf \big\{ E(v) : v \in \tilde{\mathcal{E}}(\mathbb{R}^2),\ p(v) = \mathfrak{p} \big\}.$$

Néanmoins, de façon similaire a (1.40), on a que

$$\inf \big\{ E(v) : v \in \tilde{\mathcal{E}}(\mathbb{R}^2), \quad d(v) \neq 0 \big\} = 4\pi, \tag{1.48}$$

ce qui montre que des solutions avec grande énergie ne pourraient être obtenues en considérant des fonctions de degré non nul. Pour ces raisons, on étudiera la courbe minimisante

$$E^0_{\min}(\mathfrak{p}) = \inf \big\{ E(v) : v \in \tilde{\mathcal{E}}(\mathbb{R}^2),\ p(v) = \mathfrak{p},\ d(v) = 0 \big\}.$$

Au chapitre 5, on montrera les résultats suivants.

Théorème 25. *La fonction $\mathfrak{p} \to E^0_{\min}(\mathfrak{p})$ est concave, croissante et Lipschitz continue. De plus,*

$$|E^0_{\min}(\mathfrak{p}) - E^0_{\min}(\mathfrak{q})| \leq |\mathfrak{p} - \mathfrak{q}|,$$

pour tout $\mathfrak{p}, \mathfrak{q} > 0$. En particulier

$$E^0_{\min}(\mathfrak{p}) \leq \mathfrak{p}, \qquad \textit{pour tout } \mathfrak{p} > 0,$$

et l'application $\Xi(\mathfrak{p}) := \mathfrak{p} \to \mathfrak{p} - E^0_{\min}(\mathfrak{p})$ est continue, convexe et croissante sur $\mathbb{R}_+$. En particulier, il existe $\mathfrak{p}_0 \geq 0$ tel que $\Xi(\mathfrak{p}) = 0$, pour tout $\mathfrak{p} \leq \mathfrak{p}_0$. De plus, si $0 < \mathfrak{p} < \mathfrak{p}_0$, alors $E^0_{\min}(\mathfrak{p}) = \mathfrak{p}$ et l'infimum n'est pas atteint.

Proposition 26. *Soit $\mathfrak{p} > 0$. Supposons que $E^0_{\min}(\mathfrak{p})$ est atteint pour une fonction $u_{\mathfrak{p}}$. Alors $u_{\mathfrak{p}}$ est une solution régulière de l'équation* (OPLL$_c$) *à vitesse $c = c(u_{\mathfrak{p}})$ qui vérifie*

$$0 \leq \frac{d^+}{dp}\big(E^0_{\min}(\mathfrak{p})\big) \leq c(u_{\mathfrak{p}}) \leq \frac{d^-}{dp}\big(E^0_{\min}(\mathfrak{p})\big) \leq 1,$$

où d^+/dp et d^-/dp désignent les dérivées latérales.

Le théorème 25 et la proposition 26 concordent parfaitement avec les résultats dans [83] et la figure 1.5. Ainsi, ils sont un premier pas pour essayer d'établir l'existence d'ondes progressives de manière variationnelle. Un de nos résultats principaux (voir théorème 31) a pour conséquence le théorème 32 qui établit que la courbe minimisante n'atteint pas son infimum proche de $p = 0$. Cela justifie que la courbe des solutions de la figure 1.5 est loin de l'origine, une différence très importante par rapport aux solutions de l'équation de Gross–Pitaevskii en dimension deux † (voir figure 1.2).

†. mais très similaire à la situation rencontrée en dimension supérieure à trois

1.3.2 Remarques sur la régularité

Dans l'étude des propriétés générales de l'équation (OPLL_c), le premier problème est la régularité des solutions d'énergie finie. À première vue, on note que l'équation de Landau–Lifshitz est beaucoup moins régularisante que l'équation de Gross–Pitaevskii. De plus, le terme en gradient carré dans l'équation (OPLL_c) nous empêche d'invoquer les estimations habituelles de régularité elliptique. Toutefois, certains éléments de la théorie des applications harmoniques (voir par exemple [58]) peuvent être adaptés pour traiter cette équation en dimension deux et obtenir le résultat suivant :

Proposition 27. *Soient $c \geq 0$ et $u \in \mathcal{E}(\mathbb{R}^2)$ une solution de* (OPLL_c)*. Alors $u \in C^\infty(\mathbb{R}^2) \cap \tilde{\mathcal{E}}(\mathbb{R}^2)$. De plus, il existe des constantes $\varepsilon_0 > 0$ et $K(\varepsilon_0) > 0$ telles que si $E(u) \leq \varepsilon_0$, alors*

$$\|u_3\|_{L^\infty(\mathbb{R}^2)} + \|\nabla u\|^2_{L^\infty(\mathbb{R}^2)} \leq K(\varepsilon_0)(1+c)E(u)^{1/2}.$$

Cependant en dimension supérieure ou égale à trois, la question de la régularité est beaucoup plus difficile. Par exemple, si l'on considère l'équation des applications harmoniques

$$-\Delta v = |\nabla v|^2 v, \quad \text{dans } B(0,1) \subset \mathbb{R}^N, \ v \in \mathbb{S}^2, \tag{1.49}$$

on trouve que pour $N = 3$ la fonction $v(x) = x/|x| \in \mathbb{S}^2$ est une solution d'énergie finie discontinue à l'origine. De plus, T. Rivière [90] a construit des solutions d'énergie finie de (1.49) presque partout discontinues, pour tout $N \geq 3$. Pour cette raison, pour établir nos résultats en dimension $N \geq 3$, on supposera que les solutions appartiennent à l'ensemble $\mathcal{E}(\mathbb{R}^N) \cap UC(\mathbb{R}^N)$, où $UC(\mathbb{R}^N)$ dénote l'ensemble de fonctions uniformément continues. Sous cette hypothèse, un résultat classique (voir [16, 69, 64, 80]) implique que u est en fait régulière. Plus précisément, on peut établir que :

Lemme 28. *Soient $N \geq 3$, $c \geq 0$ et $u \in \mathcal{E}(\mathbb{R}^N) \cap UC(\mathbb{R}^N)$ une solution de* (OPLL_c)*. Donc $u \in C^\infty(\mathbb{R}^N) \cap \tilde{\mathcal{E}}(\mathbb{R}^N)$. De plus, si $N \in \{3,4\}$ et $c \in [0,1]$, il existe $\varepsilon_0, K, \alpha > 0$, indépendants de u et c, tel que si $E(u) \leq \varepsilon_0$, alors*

$$\|u_3\|_{L^\infty(\mathbb{R}^N)} + \|\nabla u\|_{L^\infty(\mathbb{R}^N)} \leq KE(u)^\alpha.$$

On remarque que la proposition 27 et le lemme 28 constituent une étape préliminaire vers la preuve des résultats de non existence énoncés à la sous-section suivante.

1.3.3 Résultats de non existence et comportement à l'infini

Notre résultat principal est dans le même esprit que le théorème 18. Plus précisément, on montre l'existence d'une borne inférieure pour l'énergie en dimension $N \geq 2$, ce qui implique la non existence d'ondes progressives non triviales d'énergie petite.

Théorème 29. *Soit $N \in \{3,4\}$. Il existe une constante $\mu > 0$ telle que, pour toute solution non triviale $u \in \mathcal{E}(\mathbb{R}^2) \cap UC(\mathbb{R}^2)$ de* (OPLL_c) *avec $c \in (0,1]$, on a*

$$E(u) \geq \mu.$$

En particulier, il n'existe pas de solution non triviale d'énergie petite de (OPLL_c) *de vitesse $c \in (0,1]$.*

Comme il a été remarqué dans [56], en dimension deux il n'y a pas d'onde progressive régulière d'énergie finie de vitesse $c = 0$. Plus généralement, on a la proposition suivante pour les ondes statiques.

Proposition 30. *Soient $N \geq 2$ et $u \in \mathcal{E}(\mathbb{R}^N)$ une solution de* (OPLL_c) *avec $c = 0$. On admet aussi que $u \in UC(\mathbb{R}^N)$ si $N \geq 3$. Alors u est une solution triviale.*

Dans le cas de la dimension deux, le problème de la non existence est plus délicat et pour l'établir on a besoin que l'énergie soit petite et inférieure ou égale au moment.

Théorème 31. *Soit $M \geq 0$. Il existe une constante $\kappa_M > 0$ telle que pour toute solution non triviale $u \in \mathcal{E}(\mathbb{R}^2)$ de* (OPLL_c)*, avec $c \in (0,1)$ et satisfaisant $E(u) \leq p(u) + M(1-c^2)$, on a*

$$E(u) \geq \kappa_M.$$

En particulier, en prenant $M = 0$, on déduit qu'il n'existe pas de solution non triviale de (OPLL_c) *de vitesse $c \in (0,1)$ telle que son énergie soit petite et inférieure ou égale à son moment.*

Bien que la condition $E(u) \leq p(u)$ restreigne l'ensemble des solutions pour lequel le théorème précèdent est valable, il est suffisant pour établir le théorème suivant qui établit la non existence de solutions minimisantes données par la courbe $E^0_{\min}(\mathfrak{p})$.

Théorème 32. *Soit $\kappa_0 > 0$ la constante donnée par le théorème 31 avec $M = 0$. Alors pour tout $\mathfrak{p} \in (0, \kappa_0)$, l'infimum du problème de minimisation associé à $E^0_{\min}(\mathfrak{p})$ n'est pas atteint.*

Par ailleurs, même si on ne peut pas montrer que la condition $E(u) \leq p(u) + M(1-c^2)$ est toujours satisfaite, en général on a l'estimation à priori suivante :

Proposition 33. *Soient $c \in (0,1)$ et $u \in \mathcal{E}(\mathbb{R}^2)$ une solution de* (OPLL_c)*. Alors pour tout $\varepsilon > 0$, il existe $\bar{\varepsilon} > 0$ tel que si $E(u) \leq \bar{\varepsilon}$, on a*

$$E(u) \leq (1+\varepsilon)p(u).$$

Un des points clé dans notre étude de l'équation de Landau–Lifshitz est le fait que si $\|u_3\|_{L^\infty(\mathbb{R}^N)} < 1$, alors u_3 satisfait l'équation

$$\Delta^2 u_3 - \Delta u_3 + c^2 \partial_{1,1} u_3 = -\Delta F + c\partial_1(\operatorname{div} G), \tag{1.50}$$

où $G = (G_1, \dots G_N) := u_1 \nabla u_2 - u_2 \nabla u_1$ et $F = 2e(u)u_3 + cG_1$. Dans le cas général on peut établir une équation similaire en introduisant une fonction de troncature. On note que l'opérateur différentiel

$$\Delta^2 - \Delta + c^2 \partial_{1,1}$$

est elliptique si et seulement si $c \leq 1$, ce qui montre que la valeur $c = 1$ est critique pour l'équation (OPLL_c).

Comme on le détaillera au chapitre 5, à partir de (1.50) on obtient

$$u_3 = \mathcal{L}_c * F - c \sum_{j=1}^{N} \mathcal{L}_{c,j} * G_j,$$

où

$$\widehat{\mathcal{L}}_c = \frac{|\xi|^2}{|\xi|^4 + |\xi|^2 - c^2\xi_1^2}, \quad \widehat{\mathcal{L}}_{c,j} = \frac{\xi_1\xi_j}{|\xi|^4 + |\xi|^2 - c^2\xi_1^2}. \tag{1.51}$$

Une identité similaire est vérifiée par $\nabla\theta$. En utilisant ces équations, les arguments donnés dans [15, 28, 48, 51] nous permettent d'obtenir une décroissance algébrique pour les solutions de vitesse $c \in (0,1)$.

En outre, puisque les noyaux dans (1.51) sont les mêmes que ceux qui apparaissent dans l'étude de l'équation de Gross–Pitaevskii, à condition de prouver une certaine décroissance algébrique des solutions de (OPLL$_c$) (voir corollaire 5.7.2), on peut appliquer la théorie développée dans [48] et conclure la proposition suivante.

Proposition 34. *Soient $N \geq 3$, $c \in (0,1)$ et $u \in \mathcal{E}(\mathbb{R}^N)$ une solution de* (OPLL$_c$). *On admet aussi que $u \in UC(\mathbb{R}^N)$ si $N \geq 3$. Alors il existe $R(u) > 0$ tel que*

$$|u_3(x)| + |\nabla\theta(x)| + |\nabla\check{u}(x)| \leq \frac{K(c,u)}{1+|x|^N},$$

$$|\nabla u_3(x)| + |D^2\theta(x)| + |D^2\check{u}(x)| \leq \frac{K(c,u)}{1+|x|^{N+1}},$$

$$|D^2 u_3(x)| \leq \frac{K(c,u)}{1+|x|^{N+2}},$$

pour tout $x \in B(0,R(u))^c$.

Finalement, en utilisant ces estimations et les arguments [51], on peut calculer précisément la limite à l'infini de ces solutions comme suit.

Théorème 35. *Soient $N \geq 2$, $c \in (0,1)$ et $u \in \mathcal{E}(\mathbb{R}^N)$ une solution de* (OPLL$_c$). *On admet aussi que $u \in UC(\mathbb{R}^N)$ si $N \geq 3$. Alors il existe une constante $\lambda_\infty \in \mathbb{C}$ et deux fonctions $\check{u}_\infty, u_{3,\infty} \in C(\mathbb{S}^{N-1};\mathbb{R})$ telles que*

$$|x|^{N-1}(\check{u}(x) - \lambda_\infty) - i\lambda_\infty \check{u}_\infty\left(\frac{x}{|x|}\right) \to 0,$$

$$|x|^N u_3(x) - u_{3,\infty}\left(\frac{x}{|x|}\right) \to 0,$$

uniformément lorsque $|x| \to \infty$. De plus, en supposant sans perte de généralité que $\lambda_\infty = 1$, on a

$$\check{u}_\infty(\sigma) = \frac{\alpha\sigma_1}{(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}} + \sum_{j=2}^N \frac{\beta_j\sigma_j}{(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}},$$

$$u_{3,\infty}(\sigma) = \alpha c\left(\frac{1}{(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}} - \frac{N\sigma_1^2}{(1-c^2+c^2\sigma_1^2)^{\frac{N+2}{2}}}\right) - Nc\sum_{j=2}^N \beta_j \frac{\sigma_1\sigma_j}{(1-c^2+c^2\sigma_1^2)^{\frac{N+2}{2}}},$$

pour tout $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}$, où

$$\alpha = \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}(1-c^2)^{\frac{N-3}{2}}\left(2c\int_{\mathbb{R}^N} e(u)u_3\,dx - (1-c^2)\int_{\mathbb{R}^N} G_1(x)\,dx\right),$$

$$\beta_j = -\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}(1-c^2)^{\frac{N-1}{2}}\int_{\mathbb{R}^N} G_j(x)\,dx.$$

Nonexistence for travelling waves with small energy for the Gross–Pitaevskii equation in dimension $N \geq 3$

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Chapter 2

Nonexistence for travelling waves with small energy for the Gross–Pitaevskii equation in dimension $N \geq 3$

Abstract

We prove that the Ginzburg–Landau energy of non-constant travelling waves of the Gross–Pitaevskii equation has a lower positive bound, depending only on the dimension, in any dimension larger or equal to three. In particular, we conclude that there are no nonconstant travelling waves with small energy.

Résumé

Non existence pour les ondes progressives d'énergie petite pour l'équation de Gross–Pitaevskii en dimension $N \geq 3$. On démontre que l'énergie de Ginzburg–Landau des ondes progressives non constantes de l'équation de Gross–Pitaevskii est bornée inférieurement par une constante positive qui ne dépend que de la dimension, pour toute dimension supérieure ou égale à trois. En particulier, on en déduit qu'il n'existe pas d'onde progressive non constante d'énergie petite.

2.1 Version française abrégée

On s'intéresse aux ondes progressives non constantes d'énergie finie pour l'équation de Gross-Pitaevskii

$$i\partial_t \Psi = \Delta\Psi + \Psi(1 - |\Psi|^2) \text{ dans } \mathbb{R}^N \times \mathbb{R},$$

en dimension $N \geq 3$. Les ondes progressives pour cette équation sont des solutions de la forme

$$\Psi(x,t) = v(x_1 - ct, x_\perp), \quad x_\perp = (x_2, \ldots, x_N),$$

où la fonction v vérifie l'équation

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0 \text{ dans } \mathbb{R}^N. \tag{2.1.1}$$

Grâce aux résultats de Gravejat [47], on peut supposer que la vitesse c de l'onde progressive est telle que $0 < c \leq \sqrt{2}$. Le Hamiltonien associé à (2.1.1) est l'énergie de Ginzburg-Landau donnée par

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|v|^2)^2.$$

Tarquini a montré dans [99] l'existence d'une valeur minimale $\mathcal{E}(N,c)$ pour l'énergie de Ginzburg-Landau des ondes progressives, qui ne dépend que de la dimension N et de la vitesse c, ce qui implique que les seules solutions possibles de (2.1.1) de vitesse c avec une énergie plus petite que $\mathcal{E}(N,c)$ sont les constantes. Béthuel, Gravejat et Saut [8] ont amélioré ce résultat en dimension trois, en démontrant qu'il existe une énergie minimale $\bar{\mathcal{E}}$ indépendante de c. Dans cet article on montre qu'il est possible d'étendre ce dernier résultat pour toute dimension $N \geq 3$. Plus précisément,

Théorème 2.1.1. *Soit $N \geq 3$. Il existe une constante positive $\mathcal{E}(N)$, qui ne dépend que de N, telle que pour toute solution non constante v de* (2.1.1)*, on ait*

$$E(v) \geq \mathcal{E}(N).$$

En particulier, il n'existe pas de solution non constante pour (2.1.1) *d'énergie petite.*

2.2 Introduction

The Gross–Pitaevskii equation

$$i\partial_t \Psi = \Delta\Psi + \Psi(1-|\Psi|^2) \text{ on } \mathbb{R}^N \times \mathbb{R},$$

whose Hamiltonian is the Ginzburg–Landau energy given by

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \Psi|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|\Psi|^2)^2,$$

appears as a relevant model in several areas of physics: superfluidity, superconductivity, nonlinear optics and the Bose-Einstein condensation (see e.g. [52, 62, 61, 86]). In this work, we investigate the energy of travelling waves to this equation, i.e. solutions of the form

$$\Psi(x,t) = v(x_1 - ct, x_\perp), \quad x_\perp = (x_2, \ldots, x_N).$$

Here, the parameter $c \in \mathbb{R}$ corresponds to the speed of the travelling waves. Using complex conjugation, we may restrict to the case $c \geq 0$. The equation for the profile v is given by

$$ic\partial_1 v + \Delta v + v(1-|v|^2) = 0 \text{ on } \mathbb{R}^N. \tag{2.2.1}$$

2.3 Main result

A result of Tarquini [99] states that there exists a minimal value $\mathcal{E}(N,c)$ for the Ginzburg-Landau energy of travelling waves, depending only on N and c. This lower bound for the energy functional implies that nonconstant finite energy solutions of (2.2.1) of sufficiently small energy, with respect to their speed, are excluded in dimension $N \geq 2$. Furthermore

$$\mathcal{E}(N, c) \to 0 \quad \text{as} \quad c \to \sqrt{2}.$$

This result has been recently improved by Béthuel, Gravejat and Saut [8] in dimension three, proving that there exists some universal positive bound for the energy functional for nonconstant travelling waves.

Our aim is to extend the result of Béthuel, Gravejat and Saut [8] in any dimension larger than three, and therefore also to improve the nonexistence theorem of Tarquini [99]. More precisely, our main result is

Theorem 2.3.1. *Let $N \geq 3$. There exists some positive constant $\mathcal{E}(N)$, depending only on N, such that any nonconstant finite energy solution v of* (2.2.1) *satisfies*

$$E(v) \geq \mathcal{E}(N).$$

In particular, there are no nonconstant solutions of (2.2.1) *with small energy.*

2.4 Proof of main result

In dimension $N \geq 3$, it follows from [47] that the speed of nonconstant finite energy solutions of (2.2.1) satisfy $0 < c \leq \sqrt{2}$. From Lemma 3 in [99], we deduce that

$$\|1 - |v|^2\|_{L^\infty(\mathbb{R}^N)} \leq K(N) E(v)^{\frac{1}{2(N+1)}},$$

where $K(N)$ is a positive constant, depending only on N. Therefore, choosing a possibly smaller constant $\mathcal{E}(N)$, we may assume that v satisfies

$$\inf\{|v(x)|, x \in \mathbb{R}^N\} \geq \frac{1}{2}. \tag{2.4.1}$$

We recall that v is a smooth function (see e.g. [35]), and then in view of (2.4.1), v may be expressed as $v = \rho e^{i\varphi}$, where ρ and φ are scalar functions, and φ is defined modulo a multiple of 2π. Defining also the quantity $\eta = 1 - \rho^2$, we have

$$\Delta^2 \eta - 2\Delta\eta + c^2 \partial_1^2 \eta = -2\Delta(|\nabla v|^2 + \eta^2 - c\eta\partial_1\varphi) - 2c\partial_1 \operatorname{div}(\eta\nabla\varphi). \tag{2.4.2}$$

Applying the Fourier transform to (2.4.2), we obtain

$$\widehat{\eta}(\xi) = L_c(\xi)\widehat{F}(\xi), \tag{2.4.3}$$

where

$$\widehat{F}(\xi) = 2\widehat{R_0}(\xi) - 2c\sum_{j=2}^{N} \frac{\xi_j^2}{|\xi|^2}\widehat{R_1}(\xi) + 2c\sum_{j=2}^{N} \frac{\xi_1\xi_j}{|\xi|^2}\widehat{R_j}(\xi), \tag{2.4.4}$$

$R_0 = |\nabla v|^2 + \eta^2$, $R_j = \eta\partial_j\varphi$, $j \in \{1, \ldots, N\}$, and

$$L_c(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}. \tag{2.4.5}$$

Now we recall two results of Béthuel, Gravejat and Saut. The first one corresponds to Lemma 2.9 in [8], and the second one is an immediate extension to $\mathbb{R}^N$ of some part of the argument used in Lemma 2.15 (see inequality (2.65) in [8]).

Lemma 2.4.1. *Let v be a nonconstant finite energy solution to* (2.2.1) *satisfying* (2.4.1). *Then,*

$$E(v) \leq 7c^2 \|\eta\|^2_{L^2(\mathbb{R}^N)}.$$

Lemma 2.4.2. *For any $1 < q < \infty$, there exists a positive constant $K(N, q)$, depending only on N and q, such that*

$$\|F\|_{L^q(\mathbb{R}^N)} \leq K(N, q) E(v)^{\frac{1}{q}}.$$

We denote $\mathcal{L}_c$ the operator given by

$$\widehat{\mathcal{L}_c(f)} = L_c \widehat{f}, \quad \forall f \in S(\mathbb{R}^N).$$

We recall that in the case that there exists a constant K such that

$$\|\mathcal{L}_c(f)\|_{L^q(\mathbb{R}^N)} \leq K\|f\|_{L^p(\mathbb{R}^N)},$$

L_c is called a Fourier multiplier from L^p to L^q. We notice that identity (2.4.3) implies that η is the value of the multiplier operator associated to L_c, evaluated in the function F given by (2.4.4), that is

$$\mathcal{L}_c(F) = \eta. \tag{2.4.6}$$

In order to complete the proof of Theorem 2.3.1, we need the following lemma, whose proof we postpone to the next section.

Lemma 2.4.3. *Let $c \in (0, \sqrt{2}]$. For any $\frac{2}{2N-1} \leq \alpha \leq \frac{2}{N}$ and $\frac{1}{1-\alpha} < q < \infty$, L_c given by* (2.4.5) *is a Fourier multiplier from L^p to L^q, with $\frac{1}{p} = \frac{1}{q} + \alpha$. More precisely, there exists a positive constant $K(N, \alpha, q)$, depending only on N, α and q, such that*

$$\|\mathcal{L}_c(f)\|_{L^q(\mathbb{R}^N)} \leq K(N, \alpha, q)\|f\|_{L^p(\mathbb{R}^N)}, \ \forall f \in L^p(\mathbb{R}^N). \tag{2.4.7}$$

In view of (2.4.6), applying Lemma 2.4.3, with

$$\alpha = \frac{2}{2N-1} \quad \text{and} \quad q = 2,$$

we deduce that there exists a positive constant $K(N)$, depending only on N, such that

$$\|\eta\|_{L^2(\mathbb{R}^N)} \leq K(N)\|F\|_{L^{\frac{2(2N-1)}{2N+3}}(\mathbb{R}^N)}. \tag{2.4.8}$$

Combining Lemma 2.4.1, Lemma 2.4.2 and (2.4.8), we conclude that

$$E(v) \leq 7c^2\|\eta\|^2_{L^2(\mathbb{R}^N)} \leq 7c^2 K(N) E(v)^{\frac{2N+3}{2N-1}}. \tag{2.4.9}$$

Since $c \in (0, \sqrt{2}]$, inequality (2.4.9) implies that $E(v) \geq (14K(N))^{\frac{1-2N}{4}}$, which finishes the proof of Theorem 2.3.1.

2.5 Proof of Lemma 2.4.3

Here we use the standard multi-index notation, i.e. if $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$, $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ then $D^k = \partial_{\xi_1}^{k_1} \cdots \partial_{\xi_N}^{k_N}$, $|k| = \sum_{j=1}^N k_j$ and $\xi^k = \prod_{j=1}^N \xi_j^{k_j}$.

Lemma 2.5.1. *Let $c \in (0, \sqrt{2}]$. For any $k = (k_1, \ldots, k_N) \in \{0,1\}^N$, $m = |k|$, $1 \le m \le N$, L_c is a smooth function on $\mathbb{R}^N \backslash \{0\}$ and*

$$D^k L_c(\xi) = \frac{\xi^k}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^{m+1}} P_{m,c}(|\xi|^2, \xi_1^2), \tag{2.5.1}$$

where $P_{m,c}$ is a two-variable polynomial of degree $m+1$. More precisely, for $x, y \in \mathbb{R}$,

$$P_{m,c}(x,y) = \gamma_m(c)x^{m+1} + \sum_{\substack{i,j=0 \\ 1 \le i+j \le m}}^{m} \gamma_{m,i,j}(c)x^i y^j, \tag{2.5.2}$$

where $\{\gamma_{m,i,j}\}_{i,j=1}^m$ and γ_m are polynomial functions of the variable c. Furthermore, in the case $k_1 = 1$, setting $\alpha_m = \gamma_{m,1,0}$, $\beta_m = \gamma_{m,0,1}$ and

$$\lambda_m(c) = \frac{\alpha_m(c) + \beta_m(c)}{2 - c^2},$$

we have

$$\alpha_m(c) = (-1)^{m+1}2^{2m-1}(m-1)!c^2, \tag{2.5.3}$$

$$\beta_m(c) = (-1)^{m+1}2^{2m-2}(m-1)!c^2\left(c^2(n-1) - 2n\right), \tag{2.5.4}$$

$$\lambda_m(c) = (-1)^{m+1}2^{2m-2}(m-1)!(m-1)c^2. \tag{2.5.5}$$

In particular, λ_m is a well defined and bounded function on $(0, \sqrt{2})$.

Proof. The differentiability of L_c is immediate. The case $m = 1$ is checked explicitly, since we have

$$\partial_i L_c(\xi) = \frac{2\xi_i}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^2}\Big(- |\xi|^4 - c^2\xi_1^2 + c^2\delta_{1,i}|\xi|^2\Big). \tag{2.5.6}$$

We fix now m, with $1 < m \le N$. Let us suppose that (2.5.1) and (2.5.2) are valid for some $1 \le n < m$. We take any $r = (r_1, \ldots, r_N) \in \{0,1\}^N$ such that $|r| = n+1$ and define $j^* = \max\{1 \le j \le N \mid r_j = 1\}$. Then $j^* > 1$, and we consider $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_N) \in \{0,1\}^N$ given by $\tilde{r}_i = r_j(1 - \delta_{i,j^*})$, $i, j \in \{1, \ldots, N\}$. Therefore, $|\tilde{r}| = n$ and we have,

$$D^r L_c(\xi) = \partial_{j^*}^1 \left(\partial_1^{\tilde{r}_1}\partial_2^{\tilde{r}_2} \ldots \partial_N^{\tilde{r}_N} L_c\right)(\xi) = \frac{\xi^r}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^{n+2}} P_{n+1,c}(|\xi|^2, \xi_1^2), \tag{2.5.7}$$

where

$$P_{n+1,c}(|\xi|^2, \xi_1^2) = 2\partial_x P_{n,c}(|\xi|^2, \xi_1^2)(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2) - (n+1)(4|\xi|^2 + 4)P_{n,c}(|\xi|^2, \xi_1^2).$$

Using this inductive argument, we conclude the first part of the lemma, that is, identities (2.5.1) and (2.5.2). In order to deduce, in the case $k_1 = 1$, that the coefficients of lower terms are explicitly given by (2.5.3) and (2.5.4), we use the same inductive argument but we replace the polynomial expression (2.5.7) by the following one

$$P_{n+1,c}(x,y) = \bar{\gamma}_n(c)x^{n+2} + \sum_{\substack{i,j=0 \\ 2\leq i+j\leq n}}^{n+1} \bar{\gamma}_{n,i,j}(c)x^i y^j - 4n\alpha_n(c)x - (2c^2\alpha_n(c) + 4(n+1)\beta_n(c))y,$$

for some $\{\bar{\gamma}_{n,i,j}\}_{i,j=1}^{n}$, $\bar{\gamma}_n$, polynomial functions of the variable c. The formulas (2.5.3) and (2.5.4) allow us to finish the induction. Finally we notice that identity (2.5.5) is an immediate consequence of (2.5.3) and (2.5.4). □

An important property that follows from identities (2.5.3)-(2.5.5) is that for small values of ξ, we may compute an explicit bound for $P_{m,c}$, that is

Lemma 2.5.2. *For any $c \in [0, \sqrt{2}]$ and $0 < |\xi| \leq 1$, $k = (1, k_2, \dots, k_N)$, $m = |k|$, we have*

$$|P_{m,c}(|\xi|^2, \xi_1^2)| \leq K(N)(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2),$$

where $K(N)$ is a positive constant depending only on N.

Proof. The only delicate terms of $P_{m,c}$ to estimate are the ones associated to $|\xi|^2$ and ξ_1^2, this is $\alpha_m(c)|\xi|^2 + \beta_m(c)\xi_1^2$. Indeed, the other terms of $P_{m,c}(|\xi|^2, \xi_1^2)$ are easily bounded by $K(N)|\xi|^4$, for some constant $K(N)$ depending only on N. For example,

$$|\gamma_{m,1,1}(c)|\xi_1^2|\xi|^2 \leq \frac{1}{2}\|\gamma_{m,1,1}\|_{L^\infty[0,\sqrt{2}]}(\xi_1^4 + |\xi|^4) \leq K(N)|\xi|^4 \leq K(N)(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2),$$

where we used that the L^∞-norm in $[0, \sqrt{2}]$ of the functions $\gamma_{m,i,j}$ only depends on the dimension. Next we derive the bound for $\alpha_m(c)|\xi|^2 + \beta_m(c)\xi_1^2$. Denoting $\xi = r\sigma$, where $0 < r \leq 1$ and $\sigma = (\sigma_1, \sigma_\perp) \in \mathbb{S}^{N-1}$, this is equivalent to prove that

$$\exists K > 0, \forall c \in [0, \sqrt{2}],\ \forall \sigma_1 \in [0,1],\ \forall r \in (0,1],\ |\alpha_m(c) + \sigma_1^2\beta_m(c)| \leq K(r^2 + 2 - c^2\sigma_1^2). \quad (2.5.8)$$

Using the continuity of α_m and β_m, inequality (2.5.8) automatically follows from

$$\exists K > 0,\ \forall c \in [0, \sqrt{2}),\ \forall \rho \in [0,1],\ |\alpha_m(c) + \rho\beta_m(c)| \leq K(2 - c^2\rho). \quad (2.5.9)$$

We shall prove (2.5.9) arguing by contradiction. If (2.5.9) were false, there would exist sequences

$$K_n \to \infty,\ c_n \in [0, \sqrt{2}),\ c_n \to \bar{c} \in [0, \sqrt{2}],\ \rho_n \to \bar{\rho} \in [0,1], \quad (2.5.10)$$

such that

$$|\alpha_m(c_n) + \rho_n\beta_m(c_n)| > K_n(2 - c_n^2\rho_n) \geq 0. \quad (2.5.11)$$

In particular,

$$\lim_{n\to\infty} \frac{2 - c_n^2\rho_n}{|\alpha_m(c_n) + \rho_n\beta_m(c_n)|} = 0. \quad (2.5.12)$$

From the continuity of α_m and β_m, the denominator in (2.5.12) is bounded, so that (2.5.12) implies $\bar{c}^2\bar{\rho} = 2$, and hence $\bar{c} = \sqrt{2}$ and $\bar{\rho} = 1$. Setting $\varepsilon_n = 1 - \rho_n$ and $s_n = \frac{\varepsilon_n}{2-c_n^2}$, we write

$$\frac{2 - c_n^2\rho_n}{|\alpha_m(c_n) + \rho_n\beta_m(c_n)|} = \frac{1 + s_n c_n^2}{|\lambda_m(c_n) - s_n\beta_m(c_n)|}. \tag{2.5.13}$$

Passing possibly to a subsequence, $s_n \to \bar{s}$, with $\bar{s} \in [0, \infty]$. We note from Lemma 2.5.1 that β_m and λ_m are bounded functions of c. If $\bar{s} \in [0, \infty)$, we take the limit in (2.5.13), so that in view of (2.5.12), we deduce that $\bar{s} = -\frac{1}{2}$, which is a contradiction. We may handle the case $\bar{s} = \infty$ in a similar way, with the difference that we first divide the numerator and the denominator of the r.h.s. of (2.5.13) by s_n. Then passing to the limit, we deduce that $\bar{c} = 0$, which gives us again a contradiction. □

Now we are able to deduce a uniform bound (with respect to the speed) for L_c.

Proposition 2.5.1. *Let $c \in (0, \sqrt{2}]$ and $k = (k_1, k_2, \dots, k_N) \in \{0,1\}^N$, with $|k| \le N$. Then for any $|\xi| \ge 1$,*

$$|D^k L_c(\xi)| \le \frac{K(N)}{|\xi|^{|k|+2}}, \tag{2.5.14}$$

and for any $0 < |\xi| \le 1$,

$$|D^k L_c(\xi)| \le \frac{K(N)|\xi^k|}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^{|k|+1}} \left((1 - k_1)|\xi|^2 + k_1(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)\right), \tag{2.5.15}$$

where $K(N)$ is a constant depending only on N.

Proof. From (2.5.1) and (2.5.2), with $m = |k|$, we conclude that for any $|\xi| \ge 1$,

$$|D^k L_c(\xi)| \le K(N)|\xi|^{-3m-4}|P_{m,c}(|\xi|^2, \xi_1^2)| \le K(N)|\xi|^{-3m-4}|\xi|^{2(m+1)},$$

which proves (2.5.14). To derive (2.5.15), we note that in view of (2.5.1), it is enough to prove that for any $0 < |\xi| \le 1$,

$$|P_{m,c}(|\xi|^2, \xi_1^2)| \le K(N)((1 - k_1)|\xi|^2 + k_1(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)). \tag{2.5.16}$$

If $k_1 = 0$, inequality (2.5.16) is trivial. In the case $k_1 = 1$, this bound corresponds exactly to Lemma 2.5.2. □

Proof of Lemma 2.4.3. Firstly, we notice that the condition $N \ge 3$ implies $0 < \alpha < 1$, so that the set of valid pairs $p \ge 1$ and $q \ge 1$ is not empty. From Proposition 2.5.1 we conclude that, for any $|\xi| \ge 1$, $k = (k_1, \dots, k_N) \in \{0,1\}^N$, $|k| \le N$,

$$\prod_{j=1}^{N} |\xi_j|^{\alpha + k_j} \left| D^k L_c(\xi) \right| \le \frac{K(N)}{|\xi|^{2-N\alpha}} \le K(N), \tag{2.5.17}$$

provided that $\alpha \leq \frac{2}{N}$, for some constant $K(N)$ depending only on N. On the other hand, if $0 < |\xi| \leq 1$, we set $\xi = r\sigma$, with $r > 0$ and $\sigma = (\sigma_1, \sigma_\perp) \in \mathbb{S}^{N-1}$. Then we have that $|\xi_j| \leq r|\sigma_\perp|$, for any $j \in \{2, \dots, N\}$, and also that

$$|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2 \geq r^2(r^2 + 2\sigma_\perp^2),$$

for any $c \in (0, \sqrt{2}]$. From (2.5.15), we conclude that

$$\begin{aligned}\prod_{j=1}^{N} |\xi_j|^{\alpha+k_j} \left| D^k L_c(\xi) \right| &\leq K(N) \frac{r^{2(|k|-k_1+1)+\alpha N} |\sigma_\perp|^{\alpha(N-1)+2(|k|-k_1)}}{r^{2(|k|-k_1+1)}(r^2 + 2\sigma_\perp^2)^{|k|-k_1+1}} \\ &\leq K(N) \max\{r, |\sigma_\perp|\}^{\alpha(2N-1)-2} \leq K(N), \qquad (2.5.18)\end{aligned}$$

for any $k = (k_1, \dots, k_N) \in \{0, 1\}^N$, $|k| \leq N$, on condition that $\alpha \geq \frac{2}{2N-1}$.

Finally, from (2.5.17) and (2.5.18) we have that for every $\frac{2}{2N-1} \leq \alpha \leq \frac{2}{N}$,

$$\sup\{|\xi_1^{k_1+\alpha} \cdots \xi_1^{k_N+\alpha} D^k L_c(\xi)|, \ \ \xi \in \mathbb{R}^N \backslash \{0\}, \ k \in \{0,1\}^N, |k| \leq N\} \leq K(N),$$

and therefore Lemma 2.4.3 is now an immediate consequence of Lizorkin's multiplier theorem (see e.g. [75] and Theorem A.1). $\square$

2.6 Appendix: Fourier multipliers and the Lizorkin theorem

In this appendix we recall some facts of Fourier multipliers and we give a sketch of the proof of the Lizorkin theorem.

For $m \in S'(\mathbb{R}^N)$, we define the operator $\mathcal{M}$ by

$$\widehat{\mathcal{M}(f)} = m\widehat{f}, \quad \forall f \in S(\mathbb{R}^N), \qquad (A.1)$$

where $S(\mathbb{R}^N)$ is the Schwartz space and $S'(\mathbb{R}^N)$ is the space of tempered distributions. We note that (A.1) can also be recast in the form of the convolution equation

$$\mathcal{M}f = g * f, \quad \forall f \in S(\mathbb{R}^N),$$

where $m = \widehat{g}$.

In the case that there exist $K > 0$ and $1 \leq p \leq q \leq \infty$ such that

$$\|\mathcal{M}(f)\|_{L^q(\mathbb{R}^N)} \leq K\|f\|_{L^p(\mathbb{R}^N)}, \ \forall f \in S(\mathbb{R}^N), \qquad (A.2)$$

we say that m is an (L^p, L^q)-multiplier and $\mathcal{M}$ is its associated (L^p, L^q)-multiplier operator. If $p = q$, we just say L^p-multiplier and L^p-multiplier operator, respectively. We observe that if m is an (L^p, L^q)-multiplier, with $q < \infty$, then $\mathcal{M}$ has a unique bounded extension to $L^p(\mathbb{R}^N)$, which also satisfies (A.2). The smallest constant K satisfying (A.2) will be denoted $\|\mathcal{M}\|_{p,q}$.

For example, in the case $p = q = 2$, we have that m is an L^2-multiplier if and only if $m \in L^\infty(\mathbb{R}^N)$. In fact, if m is an L^2-multiplier, then for every $f \in L^2(\mathbb{R}^N)$, $\mathcal{M}f \in L^2(\mathbb{R}^N)$ and by the Plancherel theorem,

$$\|m\widehat{f}\|_{L^2(\mathbb{R}^N)} = (2\pi)^{\frac{N}{2}}\|\mathcal{M}(f)\|_{L^2(\mathbb{R}^N)} \leq (2\pi)^{\frac{N}{2}}\|\mathcal{M}\|_{2,2}\|f\|_{L^2(\mathbb{R}^N)} = \|\mathcal{M}\|_{2,2}\|\widehat{f}\|_{L^2(\mathbb{R}^N)}.$$

Since the Fourier transform is an isomorphism in $L^2(\mathbb{R}^N)$, we conclude that

$$\|mf\|_{L^2(\mathbb{R}^N)} \leq \|\mathcal{M}\|_{2,2}\|f\|_{L^2(\mathbb{R}^N)}, \quad \forall f \in L^2(\mathbb{R}^N).$$

Therefore m belongs to $L^2(\mathbb{R}^N)$, and $|m(\xi)| \leq \|\mathcal{M}\|_{2,2}$ a.e. on $\mathbb{R}^N$. On the other hand, if $|m(\xi)| \leq C$ a.e., by the Plancherel Theorem,

$$\|\mathcal{M}(f)\|_{L^2(\mathbb{R}^N)} = \|m\widehat{f}\|_{L^2(\mathbb{R}^N)} \leq C\|\widehat{f}\|_{L^2(\mathbb{R}^N)} = C\|f\|_{L^2(\mathbb{R}^N)}$$

By definition, this means that $\|\mathcal{M}\|_{2,2} \leq C$.

As an example of multiplier, we consider the function

$$m(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}, \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{A.3}$$

From the Riesz potential theory (see e.g. [98]), we have that m is an L^p-multiplier, for any $1 < p < \infty$. This result can also be deduced from following theorem due to Lizorkin.

Theorem A.1 ([75, 76]). *Let $m \in C^N(\mathbb{R}^N \setminus \{0\})$, $1 < p \leq q < \infty$ and suppose that exists $M > 0$ such that*

$$\sup\{|\xi_1^{k_1+\alpha} \cdots \xi_N^{k_N+\alpha} D^k m(\xi)| \ : \ \xi \in \mathbb{R}^N \setminus \{0\}, \ k \in \{0,1\}^N\} \leq M, \tag{A.4}$$

where $\alpha = \frac{1}{p} - \frac{1}{q}$. Then m is an (L^p, L^q)-multiplier. Moreover, there exists a constant $K > 0$, depending only on N, p and q, such that

$$\|\mathcal{M}\|_{p,q} \leq KM.$$

For the sake of completeness, we give the sketch of the proof given in [75] and [76].

Proof. The basic idea is to decompose m in dyadic regions and then express m as the integral of a potential in each one. For this representation, we finally prove that m is a multiplier in adequate spaces.

Let us first recall some elementary facts. Given $f \in C_0^\infty(\mathbb{R}^N)$ and $n = (n_1, \ldots, n_N) \in \mathbb{Z}^N$ we set

$$U_n = \{\xi \in \mathbb{R}^N \mid 2^{n_j} < |\xi| \leq 2^{n_j+1}, \ j = 1, \ldots, N\}.$$

Then

$$f = \sum_{n \in \mathbb{Z}^N} f_n,$$

where

$$f_n(x) = \mathcal{F}^{-1}(1_n \widehat{f}). \tag{A.5}$$

Here $\mathcal{F}^{-1}$ denotes the inverse of the Fourier transform and 1_n is the characteristic function of the set U_n. For $1 < p < \infty$, we define the space $L_2^p(\mathbb{R}^N)$ as the closure of compact supported functions with respect to the norm

$$\|f\|_{L_2^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(\sum_{n \in \mathbb{Z}^N} f_n^2(x) \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

We recall that $L_2^p(\mathbb{R}^N)$ is a Banach space and that by the Littlewood-Paley theorem (Theorem A.2 below), it coincides with the space $L^p(\mathbb{R}^N)$, with equivalence of norms.

Theorem A.2 ([94]). *Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^N)$. Then there exist constants c_1 and c_2, independent of f, such that*

$$c_1\|f\|_{L^p(\mathbb{R}^N)} \leq \|f\|_{L_2^p(\mathbb{R}^N)} \leq c_2\|f\|_{L^p(\mathbb{R}^N)}.$$

Going back to the prove of Theorem A.1, the first step is to decompose the operator $\mathcal{M}$ as $\mathcal{M} = A_3A_2A_1$, acting according to the following scheme

$$L^p(\mathbb{R}^N) \xrightarrow{A_1} L_2^p(\mathbb{R}^N) \xrightarrow{A_2} L_2^q(\mathbb{R}^N), \xrightarrow{A_3} L^q(\mathbb{R}^N)$$

where for all $n \in \mathbb{Z}^N$,

$$(A_1(f))_n = \mathcal{F}^{-1}(\widehat{f}1_n),.$$
$$(A_2(\mathrm{g}))_n = \mathcal{F}^{-1}(m\widehat{g}_n1_n),$$

for all $\mathrm{g} \equiv (g_n)_{n\in\mathbb{Z}^N} \in L_2^p(\mathbb{R}^N)$, and

$$A_3(\mathrm{g}) = \mathcal{F}^{-1}\left(\sum_{n\in\mathbb{Z}^N} \widehat{g}_n\right).$$

The map A_1 is bounded by Theorem A.2. We also have that $A_1(L^p(\mathbb{R}^N))$ is a subspace of $L_2^p(\mathbb{R}^N)$ and it contains the functions $(g_n)_{n\in\mathbb{Z}^N}$ for which $\widehat{g}_n$ vanishes outside U_n. We also notice that Theorem A.2 implies the continuity of the map $A_3 : A_2A_1(L^p(\mathbb{R}^N)) \to L^q(\mathbb{R}^N)$. To prove the continuity of A_2, we assume for simplicity that $p < q$. We will give several lemmas proved in [75] that yield the conclusion and we only give some remarks about their proofs.

Lemma A.3. *For any $n \in \mathbb{Z}^N$, there exists a bounded variation function ρ_n such that*

$$m_n(\xi) \equiv m(\xi)1_n(\xi) = \int_{-\infty}^{\xi_1} \cdots \int_{-\infty}^{\xi_N} \frac{\rho_n(\eta)d\eta}{(\xi_1-\eta_1)^\alpha \cdots (\xi_N-\eta_N)^\alpha}, \tag{A.6}$$

and

$$\operatorname*{var}_{\xi\in\mathbb{R}^N} \rho_n(\xi) = \int_{\mathbb{R}^N} |\rho_n'(\xi)|d\xi \leq cM, \tag{A.7}$$

for some constant $c > 0$, independent of m, n and M.

Proof idea. The integral equation (A.6) can be explicitly solved using the Fourier transform, resulting

$$\rho_n = c\left(\prod_{j=1}^N \frac{1}{\xi_j^{1-\alpha}}\right) * \partial_{\xi_1\ldots\xi_N} m_n,$$

where the last derivate is considered in a distributional sense. From here, using (A.4), it is straightforward to verify (A.7). □

Now we explain how to obtain the bounds for some intermediate multipliers.

Lemma A.4. *Let $1 < p < q < \infty$. Then*

$$I_\alpha(f)(x) = \int_{\mathbb{R}^N} f(y) \prod_{j=1}^N \frac{1}{|x_j-y_j|^{1-\alpha}} dy$$

is an (L^p, L^q)-multiplier operator.

Proof. The case $N = 1$ is the Hardy–Littlewood–Sobolev theorem (see e.g. [98]). The general case follows from an inductive argument. $\square$

In terms of the Fourier transform, the operator I_α can be expressed as

$$\widehat{I_\alpha f}(\xi) = c(N,\alpha) \prod_{j=1}^{N} \frac{1}{|\xi_j|^\alpha} \widehat{f}(\xi), \quad \forall f \in S(\mathbb{R}^N).$$

Similarly, for $\eta = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$, we consider the operator $J_{\alpha,\eta}$ given by

$$\widehat{J_{\alpha,\eta} f}(\xi) = (\xi - \eta)_+^{-\alpha} \widehat{f}(\xi), \quad \forall f \in S(\mathbb{R}^N),$$

where $(\cdot)_+$ is the function $x_+ = \max\{0, x\}$, $\forall x \in \mathbb{R}$. Then we have

Lemma A.5. *Let $1 < p < q < \infty$, $\alpha = \frac{1}{p} - \frac{1}{q}$, $\eta \in \mathbb{R}^N$ and $f \in L^p(\mathbb{R}^N)$. Then $J_{\alpha,\eta}$ is an (L^p, L^q)-multiplier operator.*

Now we return to the terms of type (A.6).

Lemma A.6. *Let $1 < p < q < \infty$ and*

$$m(\xi) = \int_{-\infty}^{\xi_1} \cdots \int_{-\infty}^{\xi_N} \frac{\rho(\eta) d\eta}{(\xi_1 - \eta_1)_+^\alpha \cdots (\xi_N - \eta_N)_+^\alpha}, \tag{A.8}$$

where ρ is a bounded variation function, with

$$\operatorname*{var}_{\xi \in \mathbb{R}^N} \rho(\xi) = \int_{\mathbb{R}^N} |\rho(\eta)| d\eta \le M, \tag{A.9}$$

then m is an-(L^p, L^q) multiplier. Moreover the respective multiplier operator $\mathcal{M}$ satisfies

$$\|\mathcal{M}\|_{p,q} \le cM,$$

for some constant $c > 0$ independent of M.

Proof. From the definition of $\mathcal{M}$,

$$\begin{aligned}
\mathcal{M} f(x) &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} m(\xi) \widehat{f}(\xi) e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} (\xi - \eta)_+^{-\alpha} \rho(\eta) d\eta \right) \widehat{f}(\xi) e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} (\xi - \eta)_+^{-\alpha} \widehat{f}(\xi) e^{ix\cdot\xi} d\xi \right) \rho(\eta) d\eta \\
&= \int_{\mathbb{R}^N} J_{\alpha,\eta}(f)(x) \rho(\eta) d\eta.
\end{aligned}$$

Then, using the Minkowski inequality, we have

$$\|\mathcal{M} f\|_{L^q(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} \|J_{\alpha,\eta}(f)\|_{L^q(\mathbb{R}^N)} |\rho(\eta)| d\eta,$$

so that, by Lemma A.5 and (A.9),

$$\|\mathcal{M} f\|_{L^q(\mathbb{R}^N)} \le M \|J_{\alpha,\eta}\|_{p,q} \|f\|_{L^p(\mathbb{R}^N)},$$

which finishes the proof. $\square$

In the case of a function $\mathrm{f} = (f_n)_{n\in\mathbb{Z}^N}$ in $L_2^p(\mathbb{R}^N)$, the Fourier transform is understood in the sense

$$\widehat{\mathrm{f}} = (\widehat{f_n})_{n\in\mathbb{Z}^N},$$

and the convolution with a distribution $h \in S'(\mathbb{R}^N)$ as

$$h * \mathrm{f} = (h * f_n)_{n\in\mathbb{Z}^N}.$$

In this manner,

$$\widehat{h * \mathrm{f}} = \widehat{h}\widehat{\mathrm{f}} = (\widehat{h}\widehat{f_n})_{n\in\mathbb{Z}^N}.$$

and hence it is possible to give an $L_2^p - L_2^q$ version of Lemma A.6. Furthermore, it could be extended to a family of multipliers given by a family of bounded variation functions.

Lemma A.7. *Let $1 < p < q < \infty$ and*

$$m_n(\xi) = \int_{-\infty}^{\xi_1} \cdots \int_{-\infty}^{\xi_N} \frac{\rho_n(\eta)d\eta}{(\xi_1 - \eta_1)_+^\alpha \cdots (\xi_N - \eta_N)_+^\alpha}, \quad n \in \mathbb{Z}^N,$$

where ρ_n are bounded variation functions satisfying

$$\sup_{n\in\mathbb{Z}^N} \operatorname*{var}_{\xi\in\mathbb{R}^N} \rho_n(\xi) \leq M.$$

Then m_n is an (L_2^p, L_2^q)-multiplier, in the sense that the operator $\mathcal{M}_n$ defined by

$$\widehat{\mathcal{M}_n(f)} = m\widehat{f}, \quad \forall f \in L_2^p(\mathbb{R}^N)$$

satisfies

$$\|\mathcal{M}_n(f)\|_{L_2^q(\mathbb{R}^N)} \leq cM \quad \forall f \in L_2^p(\mathbb{R}^N).$$

From Lemmas A.3 and A.7, we conclude that A_2 is continous and then the proof of Theorem A.1 is completed. $\square$

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Chapter 3

Global well-posedness for a nonlocal Gross-Pitaevskii equation with non-zero condition at infinity

Abstract

We study the Gross-Pitaevskii equation involving a nonlocal interaction potential. Our aim is to give sufficient conditions that cover a variety of nonlocal interactions such that the associated Cauchy problem is globally well-posed with non-zero boundary condition at infinity, in any dimension. We focus on even potentials that are positive definite or positive tempered distributions.
Keywords Nonlocal Schrödinger equation; Gross-Pitaevskii equation; Global well-posedness; Initial value problem.

Mathematics Subject Classification 35Q55; 35A05; 37K05; 35Q40; 81Q99.

3.1 Introduction

3.1.1 The problem

In order to describe the kinetic of a weakly interacting Bose gas of bosons of mass m, Gross [52] and Pitaevskii [86] derived in the Hartree approximation, that the wavefunction Ψ governing the condensate satisfies

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2m}\Delta\Psi(x,t) + \Psi(x,t)\int_{\mathbb{R}^N}|\Psi(y,t)|^2V(x-y)\,dy, \text{ on } \mathbb{R}^N\times\mathbb{R}, \tag{3.1.1}$$

where N is the space dimension and V describes the interaction between bosons. In the most typical first approximation, V is considered as a Dirac delta function, which leads to the standard local Gross-Pitaevskii equation. This local model with non-vanishing condition at infinity has been intensively used, due to its application in various areas of physics, such as superfluidity, nonlinear optics and Bose-Einstein condensation [62, 61, 65, 26]. It seems then natural to analyze the equation (3.1.1) for more general interactions. Indeed, in the study of superfluidity, supersolids and Bose-Einstein condensation, different types of nonlocal potentials have been proposed [6, 32, 96, 87, 63, 1, 103, 27, 23].

To obtain a dimensionless equation, we take the average energy level per unit mass $\mathcal{E}_0$ of a boson, and we set

$$\psi(x,t) = \exp\left(\frac{im\mathcal{E}_0 t}{\hbar}\right)\Psi(x,t).$$

Then (3.1.1) turns into

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\Delta\psi(x,t) - m\mathcal{E}_0\psi(x,t) + \psi(x,t)\int_{\mathbb{R}^N}|\psi(y,t)|^2 V(x-y)\,dy. \tag{3.1.2}$$

Defining the rescaling

$$u(x,t) = \frac{1}{\lambda\sqrt{m\mathcal{E}_0}}\left(\frac{\hbar}{\sqrt{2m^2\mathcal{E}_0}}\right)^{\frac{N}{2}}\psi\left(\frac{\hbar x}{\sqrt{2m^2\mathcal{E}_0}}, \frac{\hbar t}{m\mathcal{E}_0}\right),$$

from (3.1.2) we deduce that

$$i\partial_t u(x,t) + \Delta u(x,t) + u(x,t)\left(1 - \lambda^2\int_{\mathbb{R}^N}|u(y,t)|^2\mathcal{V}(x-y)\,dy\right) = 0,$$

with

$$\mathcal{V}(x) = V\left(\frac{\hbar x}{\sqrt{2m^2\mathcal{E}_0}}\right).$$

If we assume that the convolution between $\mathcal{V}$ and a constant is well-defined and equal to a positive constant, choosing $\lambda^2 = (\mathcal{V} * 1)^{-1}$, equation (3.1.2) is equivalent to

$$i\partial_t u + \Delta u + \lambda^2 u(\mathcal{V} * (1-|u|^2)) = 0 \text{ on } \mathbb{R}^N \times \mathbb{R}. \tag{3.1.3}$$

More generally, we consider the Cauchy problem for the nonlocal Gross-Pitaevskii equation with non-zero initial condition at infinity in the form

$$\begin{cases} i\partial_t u + \Delta u + u(W * (1-|u|^2)) = 0 \text{ on } \mathbb{R}^N \times \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{NGP}$$

where

$$|u_0(x)| \to 1, \quad \text{as} \quad |x| \to \infty. \tag{3.1.4}$$

If W is a real-valued even distribution, (NGP) is a Hamiltonian equation whose energy given by

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u(t)|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N}(W * (1-|u(t)|^2))(1-|u(t)|^2)\,dx$$

is formally conserved.

In the case that W is the Dirac delta function, (NGP) corresponds to the local Gross-Pitaevskii equation and the Cauchy problem in this instance has been studied by Béthuel and

Saut [12], Gérard [41], Gallo [39], among others. As mentioned before, in a more general framework the interaction kernel W could be nonlocal. For example, Shchesnovich and Kraenkel in [96] consider for $\varepsilon > 0$,

$$W_\varepsilon(x) = \begin{cases} \dfrac{1}{2\pi\varepsilon^2} K_0\left(\dfrac{|x|}{\varepsilon}\right), & N = 2, \\ \dfrac{1}{4\pi\varepsilon^2 |x|} \exp\left(-\dfrac{|x|}{\varepsilon}\right), & N = 3, \end{cases} \tag{3.1.5}$$

where K_0 is the modified Bessel function of second kind (also called MacDonald function). In this way W_ε might be considered as an approximation of the Dirac delta function, since $W_\varepsilon \to \delta$, as $\varepsilon \to 0$, in a distributional sense. Others interesting nonlocal interactions are the soft core potential

$$W(x) = \begin{cases} 1, & \text{if } |x| < a, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1.6}$$

with $a > 0$, which is used in [63, 1] to the study of supersolids, and also

$$W = \alpha_1 \delta + \alpha_2 K, \qquad \alpha_1, \alpha_2 \in \mathbb{R}, \tag{3.1.7}$$

where K is the singular kernel

$$K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}, \quad x \in \mathbb{R}^3 \backslash \{0\}. \tag{3.1.8}$$

The potential (3.1.7)-(3.1.8) models dipolar forces in a quantum gas (see [23], [103]).

3.1.2 Main results

In order to include interactions such as (3.1.7)-(3.1.8), it is appropriate to work in the space $\mathcal{M}_{p,q}(\mathbb{R}^N)$, that is the set of tempered distributions W such that the linear operator $f \mapsto W * f$ is bounded from $L^p(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$. We denote by $\|W\|_{p,q}$ its norm. We will suppose that there exist

$$p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4, s_1, s_2 \in [1, \infty),$$

with

$$\frac{N}{N-2} > p_4, \quad \frac{2N}{N-2} > p_2, p_3, s_1, s_2 \geq 2, \quad 2 \geq q_1 > \frac{2N}{N+2}, \quad q_3, q_4 > \frac{N}{2} \quad \text{if } N \geq 3$$

and

$$p_2, p_3, s_1, s_2 \geq 2, \quad 2 \geq q_1 > 1 \quad \text{if } 2 \geq N \geq 1,$$

such that

$$\begin{cases} W \in \mathcal{M}_{2,2}(\mathbb{R}^N) \cap \displaystyle\bigcap_{i=1}^4 \mathcal{M}_{p_i,q_i}(\mathbb{R}^N), \\ \dfrac{1}{p_3} + \dfrac{1}{q_2} = \dfrac{1}{q_1}, \quad \dfrac{1}{p_1} - \dfrac{1}{p_3} = \dfrac{1}{s_1}, \quad \dfrac{1}{q_1} - \dfrac{1}{q_3} = \dfrac{1}{s_2} \quad \text{if } N \geq 3. \end{cases} \tag{$\mathcal{W}_N$}$$

We recall that if $p > q$, then $\mathcal{M}_{p,q} = \{0\}$. Therefore if we suppose that W is not zero, the numbers above have to satisfy $q_2, q_3 \geq 2$. In addition, the existence of s_1, s_2 and the relations in $(\mathcal{W}_N)$ imply that

$$\frac{N}{N-2} > p_1, \quad q_2 > \frac{N}{2}, \quad \frac{1}{p_1} - \frac{1}{p_3} \in \left(\frac{N-2}{2N}, \frac{1}{2}\right], \quad \frac{1}{q_1} - \frac{1}{q_3} \in \left(\frac{N-2}{2N}, \frac{1}{2}\right] \quad \text{if } N \geq 3.$$

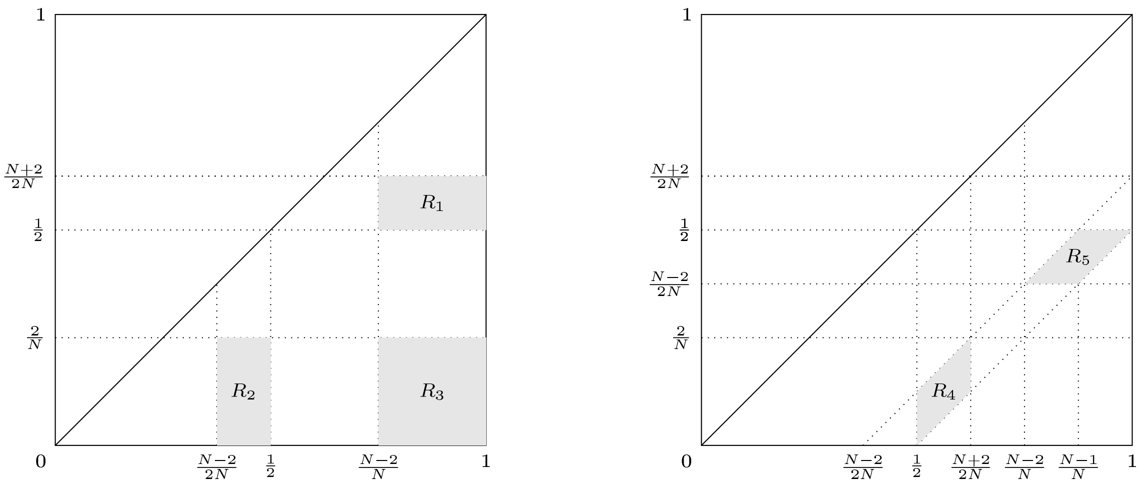

Figure 3.1: *For $N > 4$, the picture on the left represents the $(1/p, 1/q)$-plane, in the sense that $(1/p_1, 1/q_1) \in R_1$, $(1/p_2, 1/q_2), (1/p_3, 1/q_3) \in R_2$, $(1/p_4, 1/q_4) \in R_3$. In the picture on the right, the shaded areas symbolize that $(1/q_1, 1/q_3) \in R_4$ and $(1/p_1, 1/p_3) \in R_5$, for $N > 6$.*

Figure 3.1 schematically shows the location of these numbers in the unit square.

To check the hypothesis $(\mathcal{W}_N)$ it is convenient to use some properties of the spaces $\mathcal{M}_{p,q}(\mathbb{R}^N)$. For instance, for any $1 < p \leq q < \infty$, $\mathcal{M}_{p,q}(\mathbb{R}^N) = \mathcal{M}_{q',p'}(\mathbb{R}^N)$ and for any $1 \leq p \leq 2$, $\mathcal{M}_{1,1}(\mathbb{R}^N) \subseteq \mathcal{M}_{p,p}(\mathbb{R}^N) \subseteq \mathcal{M}_{2,2}(\mathbb{R}^N)$ ([46]). In Proposition 3.1.3 we give more explicit conditions to ensure $(\mathcal{W}_N)$.

As remarked before, the energy is formally conserved if W is a real-valued even distribution. We recall that a real-valued distribution is said to be even if

$$\langle W, \phi \rangle = \langle W, \widetilde{\phi} \rangle, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N; \mathbb{R}),$$

where $\widetilde{\phi}(x) = \phi(-x)$. However, the conservation of energy is not sufficient to study the long time behavior of the Cauchy problem, because the potential energy is not necessarily nonnegative and the nonlocal nature of the problem prevents us to obtain pointwise bounds. We are able to control this term assuming further that W is a *positive distribution* or supposing that it is a *positive definite distribution*. More precisely, we say that W is a positive distribution if

$$\langle W, \phi \rangle \geq 0, \quad \forall \phi \geq 0, \ \phi \in C_0^\infty(\mathbb{R}^N; \mathbb{R}),$$

and that it is a positive definite distribution if

$$\langle W, \phi * \widetilde{\phi} \rangle \geq 0, \quad \phi \in C_0^\infty(\mathbb{R}^N; \mathbb{R}). \tag{3.1.9}$$

These type of distributions frequently arise in the physical models (see Subsection 3.1.3). In particular, the real-valued even positive definite distributions include a large variety of models where the interaction between particles is symmetric. In Section 3.2 we state further properties of this kind of potentials.

As Gallo in [39], we consider the initial data u_0 for the problem (NGP) belonging to the space $\phi + H^1(\mathbb{R}^N)$, with ϕ a function of finite energy. More precisely, from now on we assume that ϕ is a complex-valued function that satisfies

$$\phi \in W^{1,\infty}(\mathbb{R}^N),\ \nabla\phi \in H^2(\mathbb{R}^N) \cap C(B^c),\ |\phi|^2 - 1 \in L^2(\mathbb{R}^N), \tag{3.1.10}$$

where B^c denotes the complement of some ball $B \subseteq \mathbb{R}^N$, so that in particular ϕ satisfies (3.1.4).

Remark 3.1.1. We do not suppose that ϕ has a limit at infinity. In dimensions $N = 1, 2$ a function satisfying (3.1.10) could have complicated oscillations, such as (see [41, 40])

$$\phi(x) = \exp(i(\ln(2 + |x|))^{\frac{1}{4}}), \quad x \in \mathbb{R}^2.$$

We note that any function verifying (3.1.10) belongs to the Homogeneous Sobolev space

$$\dot{H}^1(\mathbb{R}^N) = \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^N) : \nabla\psi \in L^2(\mathbb{R}^N)\}.$$

In particular, if $N \geq 3$ there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $\phi - z_0 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ (see e.g. Theorem 4.5.9 in [59]). Choosing $\alpha \in \mathbb{R}$ such that $z_0 = e^{i\alpha}$ and since the equation (NGP) is invariant by a phase change, one can assume that $\phi - 1 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, but we do not use explicitly this decay in order to handle at the same time the two-dimensional case.

Our main result concerning the global well-posedness for the Cauchy problem is the following.

Theorem 3.1.2. *Let W be a real-valued even distribution satisfying $(\mathcal{W}_N)$.*

(i) Assume that one of the following is verified

(a) $N \geq 2$ and W is a positive definite distribution.

(b) $N \geq 1$, $W \in \mathcal{M}_{1,1}(\mathbb{R}^N)$ and W is a positive distribution.

Then the Cauchy problem (NGP) *is globally well-posed in $\phi + H^1(\mathbb{R}^N)$. More precisely, for every $w_0 \in H^1(\mathbb{R}^N)$ there exists a unique $w \in C(\mathbb{R}, H^1(\mathbb{R}^N))$, for which $\phi + w$ solves* (NGP) *with the initial condition $u_0 = \phi + w_0$ and for any bounded closed interval $I \subset \mathbb{R}$, the flow map $w_0 \in H^1(\mathbb{R}^N) \mapsto w \in C(I, H^1(\mathbb{R}^N))$ is continuous. Furthermore, $w \in C^1(\mathbb{R}, H^{-1}(\mathbb{R}^N))$ and the energy is conserved*

$$E_0 := E(\phi + w_0) = E(\phi + w(t)),\ \forall t \in \mathbb{R}. \tag{3.1.11}$$

(ii) Assume that there exists $\sigma > 0$ such that

$$\operatorname{ess\,inf} \widehat{W} \geq \sigma. \tag{3.1.12}$$

Then (NGP) *is globally well-posed in $\phi + H^1(\mathbb{R}^N)$, for all $N \geq 1$ and* (3.1.11) *holds. Moreover, if u is the solution associated to the initial data $u_0 \in \phi + H^1(\mathbb{R}^N)$, we have the growth estimate*

$$\|u(t) - \phi\|_{L^2} \leq C|t| + \|u_0 - \phi\|_{L^2}, \tag{3.1.13}$$

for any $t \in \mathbb{R}$, where C is a positive constant that depends only on E_0, W, ϕ and σ.

We make now some remarks about Theorem 3.1.2.

- The condition $(\mathcal{W}_N)$ implies that $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$, so that $\widehat{W} \in L^\infty(\mathbb{R}^N)$ and therefore the condition (3.1.12) makes sense.

– In contrast with (3.1.13), as we prove in Section 3.5, the growth estimate for the solution given by Theorem 3.1.2-(i) is only exponential

$$\|u(t) - \phi\|_{L^2} \le C_1 e^{C_2|t|}(1 + \|u_0 - \phi\|_{L^2}), \qquad t \in \mathbb{R},$$

for some constants C_1, C_2 only depending on E_0, W and ϕ.

– Accordingly to Remark 3.1.1 and the Sobolev embedding theorem, after a phase change independent of t, the solution u of (NGP) given by Theorem 3.1.2 also satisfies that $u - 1 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ if $N \ge 3$.

– In dimensions $1 \le N \le 3$ we can choose $(p_4, q_4) = (2, 2)$ in $(\mathcal{W}_N)$. Consequently, the condition that $W \in \mathcal{M}_{p_4,q_4}(\mathbb{R}^N)$ is nontrivial only when $N \ge 4$.

At first sight, it is not obvious to check the hypotheses on W. The purpose of the next result is to give sufficient conditions to ensure $(\mathcal{W}_N)$.

Proposition 3.1.3.

(i) Let $1 \le N \le 3$. If $W \in \mathcal{M}_{2,2}(\mathbb{R}^N) \cap \mathcal{M}_{3,3}(\mathbb{R}^N)$, then W fulfils $(\mathcal{W}_N)$. Furthermore, if W verifies $(\mathcal{W}_N)$ with $p_i = q_i$, $1 \le i \le 3$, then $W \in \mathcal{M}_{2,2}(\mathbb{R}^N) \cap \mathcal{M}_{3,3}(\mathbb{R}^N)$.

(ii) Let $N \ge 4$. Assume that $W \in \mathcal{M}_{r,r}(\mathbb{R}^N)$ for every $1 < r < \infty$. Also suppose that there exists $\bar{r} > \frac{N}{4}$ such that $W \in \mathcal{M}_{p,q}(\mathbb{R}^N)$, for every $1 - \frac{1}{\bar{r}} < \frac{1}{p} < 1$ with $\frac{1}{q} = \frac{1}{p} + \frac{1}{\bar{r}} - 1$. Then W satisfies $(\mathcal{W}_N)$.

We conclude from Proposition 3.1.3 that the Dirac delta function verifies $(\mathcal{W}_N)$ in dimensions $1 \le N \le 3$. Since $\widehat{\delta} = 1$, Theorem 3.1.2-(ii) recovers the results of global existence for the local Gross-Pitaevskii equation in [12, 41, 39] and the growth estimate proved in [2]. In addition, if the potential converges to the Dirac delta function, the correspondent solutions converge to the solution of the local problem as a consequence of the following result.

Proposition 3.1.4. *Assume that $1 \le N \le 3$. Let $(W_n)_{n \in \mathbb{N}}$ be a sequence of real-valued distributions in $\mathcal{M}_{2,2}(\mathbb{R}^N) \cap \mathcal{M}_{3,3}(\mathbb{R}^N)$ such that u_n is the global solution of* (NGP) *given by Theorem 3.1.2, with W_n instead of W, for some initial data in $\phi + H^1(\mathbb{R}^N)$, and*

$$\lim_{n \to \infty} W_n = W_\infty, \quad \text{in } \mathcal{M}_{2,2}(\mathbb{R}^N) \cap \mathcal{M}_{3,3}(\mathbb{R}^N), \tag{3.1.14}$$

with $\|W_\infty\|_{\mathcal{M}_{2,2} \cap \mathcal{M}_{3,3}} > 0$ ($\|\cdot\|_{\mathcal{M}_{2,2} \cap \mathcal{M}_{3,3}} := \max\{\|\cdot\|_{\mathcal{M}_{2,2}}, \|\cdot\|_{\mathcal{M}_{3,3}}\}$). Then $u_n \to u$ in $C(I, H^1(\mathbb{R}^N))$, for any bounded closed interval $I \subset \mathbb{R}$, where u is the solution of (NGP) *with $W = W_\infty$ and the same initial data.*

On the other hand, the Dirac delta function does not satisfy $(\mathcal{W}_N)$ if $N \ge 4$ and therefore Theorem 3.1.2 cannot be applied. In fact, to our knowledge there is no proof for the global well-posedness to the local Gross-Pitaevskii equation in dimension $N \ge 4$ with arbitrary initial condition. For small initial data, Gustafson et al. [54] proved global well-posedness in dimensions $N \ge 4$ as well as Gérard [41] in the four-dimensional energy space.

As a consequence of Theorem 3.1.2 and Proposition 3.1.3 we derive the next result for integrable kernels.

Corollary 3.1.5. *Let W be a real-valued even function such that $W \in L^1(\mathbb{R}^N)$ if $1 \le N \le 3$ and $W \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, for some $r > \frac{N}{4}$, if $N \ge 4$. Assume also that W is positive definite if $N \ge 2$, or that it is nonnegative. Then the Cauchy problem* (NGP) *is globally well-posed in $\phi + H^1(\mathbb{R}^N)$.*

As Gallo remarks in [39], the well-posedness in a space such as $\phi + H^1(\mathbb{R}^N)$ makes possible to handle the problem with initial data in the energy space

$$\mathcal{E}(\mathbb{R}^N) = \{u \in H^1_{\text{loc}}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N),\ 1 - |u|^2 \in L^2(\mathbb{R}^N)\},$$

equipped with the distance

$$d(u,v) = \|u - v\|_{X^1+H^1} + \||u|^2 - |v|^2\|_{L^2}. \tag{3.1.15}$$

Here $X^1(\mathbb{R}^N)$ denotes the Zhidkov space

$$X^1(\mathbb{R}^N) = \{u \in L^\infty(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}.$$

We recall that $u \in C(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ is called a *mild solution* of (NGP) if it satisfies the Duhamel formula

$$u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}(u(s)(W * (1 - |u(s)|^2))\,ds, \quad t \in \mathbb{R}.$$

We note that by Lemma 3.6.3 the integral in the r.h.s is actually finite (see [41, 40] for further results about the action of Schrödinger semigroup on $\mathcal{E}(\mathbb{R}^N)$). With the same arguments of [39], we may also handle the problem with initial data in the energy space. Moreover, in the case $1 \leq N \leq 4$, we prove that a solution in the energy space with initial condition $u_0 \in \mathcal{E}(\mathbb{R}^N)$, necessarily belongs to $u_0 + H^1(\mathbb{R}^N)$, which is a proper subset of $\mathcal{E}(\mathbb{R}^N)$. This also gives the uniqueness in the energy space for $1 \leq N \leq 4$, as follows.

Theorem 3.1.6. *Let W be as in Theorem 3.1.2. Then for any $u_0 \in \mathcal{E}(\mathbb{R}^N)$, there exists a unique $w \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ such that $u := u_0 + w$ solves* (NGP)*. Furthermore, if $1 \leq N \leq 4$ and $v \in C(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ is a mild solution of* (NGP) *with $v(0) = u_0$, then $v = u$.*

The next proposition shows that the hypotheses made on the potential W also ensure the H^2-regularity of the solutions.

Proposition 3.1.7. *Let W be as in Theorem 3.1.2 and u be the global solution of* (NGP) *for some initial data $u_0 \in \phi + H^2(\mathbb{R}^N)$. Then $u - \phi \in C(\mathbb{R}, H^2(\mathbb{R}^N)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N))$.*

Finally, we study the conservation of momentum and mass for (NGP). As has been discussed in several works (see [7, 9, 77, 10]) the classical concepts of momentum and mass, that is

$$p(u) = \int_{\mathbb{R}^N} \langle\langle i\nabla u, u\rangle\rangle\,dx \quad \text{and} \quad M(u) = \int_{\mathbb{R}^N} (1 - |u|^2)\,dx,$$

with $\langle\langle z_1, z_2\rangle\rangle = \operatorname{Re}(z_1\overline{z}_2)$, are not well-defined for $u \in \phi + H^1(\mathbb{R}^N)$. Thus it is necessary to give some generalized sense to these quantities. In Section 3.7 we will explain in detail a notion of *generalized momentum* and *generalized mass* such that we have the next results on conservation laws.

Theorem 3.1.8. *Let $N \geq 1$ and $u_0 \in \phi + H^1(\mathbb{R}^N)$. Then the generalized momentum is conserved by the flow of the associated solution u of* (NGP) *given by Theorem 3.1.2.*

Theorem 3.1.9. *Let $1 \leq N \leq 4$. In addition to* (3.1.10)*, assume that $\nabla\phi \in L^{\frac{N}{N-1}}(\mathbb{R}^N)$ if $N = 3, 4$. Suppose that $u_0 \in \phi + H^1(\mathbb{R}^N)$ has finite generalized mass. Then the generalized mass of the associated solution of* (NGP) *given by Theorem 3.1.2 is conserved by the flow.*

3.1.3 Examples

(i) Given the spherically symmetric interaction of bosons, it is usual to suppose that W is radial, that is $W(x-y) = R(|x-y|)$, with $R : [0,\infty) \to \mathbb{R}$. Using the fact that the Fourier transform of a radial function is also radial, we may write $\widehat{W}(\xi) = \rho(|\xi|)$, for some function $\rho : [0,\infty) \to \mathbb{R}$. Noticing that $\widehat{\delta} = 1$, a next order of approximation would be to consider (see e.g. [96])

$$\rho(r) = \frac{1}{1+\varepsilon^2 r^2}, \qquad \varepsilon > 0.$$

Then the Fourier inversion theorem implies that W is given by (3.1.5) for $N = 2,3$. By Proposition 3.2.2, (3.1.5) is indeed a positive definite function, since ρ is nonnegative. For this potential we also have that $K_0(x) \approx \ln\left(\frac{2}{x}\right)$ as $x \to 0$, and $K_0(x) \approx \sqrt{\frac{\pi}{2x}} \exp(-x)$ as $x \to \infty$ (see e.g. [73], p. 136), hence $W \in L^1(\mathbb{R}^N)$ for $N = 2,3$. Therefore it is possible to invoke Corollary 3.1.5.

(ii) By Lemma 3.2.3, the function given by (3.1.6) cannot be positive definite, since it is bounded and it does not coincide with any continuous function a.e. However, W is a nonnegative function that belongs to $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Therefore Corollary 3.1.5 can be applied in any dimension.

(iii) We recall that if Ω is an even function, smooth away from the origin, homogeneous of degree zero, with zero mean-value on the sphere

$$\int_{\mathbb{S}^{N-1}} \Omega(\sigma)\, d\sigma = 0,$$

then

$$K(x) = \frac{\Omega(x)}{|x|^N}, \quad x \in \mathbb{R}^N \backslash \{0\},$$

defines a tempered distribution $\mathcal{K}$ in the sense of principal value, that coincides with K away from the origin. Moreover, for any $f \in S(\mathbb{R}^N)$, $x \in \mathbb{R}^N$,

$$(\mathcal{K} * f)(x) = \text{p.v.} \int_{\mathbb{R}^N} K(y) f(x-y)\, dy = \lim_{\varepsilon \to 0} \int_{\frac{1}{\varepsilon} > |y| > \varepsilon} \frac{\Omega(y)}{|y|^N} f(x-y)\, dy, \tag{3.1.16}$$

$\mathcal{K} \in \mathcal{M}_{p,p}(\mathbb{R}^N)$ for every $1 < p < \infty$, and the Fourier transform of $\mathcal{K}$ belongs to $L^\infty(\mathbb{R}^N)$ (cf. [98]). Therefore

$$W = \alpha_1 \delta + \alpha_2 \mathcal{K} \tag{3.1.17}$$

is a positive definite distribution if α_1 is large enough and then Theorem 3.1.2-(ii) gives a global solution of (NGP) in any dimension. For instance, we may consider in dimension three the function K given by (3.1.8). Since (see [23])

$$\widehat{\mathcal{K}}(\xi) = \frac{4\pi}{3}\left(\frac{3\xi_3^2}{|\xi|^2} - 1\right), \qquad \xi \in \mathbb{R}^3 \backslash \{0\},$$

(3.1.17) is positive definite by Proposition 3.2.2 if

$$\alpha_1 \geq \frac{4\pi}{3}\alpha_2 \geq 0 \qquad \text{or} \qquad \alpha_1 \geq -\frac{8\pi}{3}\alpha_2 \geq 0. \tag{3.1.18}$$

Therefore, if (3.1.18) is verified we may apply Theorem 3.1.2-(i)-(a). Moreover, if the inequalities in (3.1.18) are strict, we have also the growth estimate of Theorem 3.1.2-(ii).

(iv) Let us recall that to pass from the original equation (3.1.1) to (3.1.3) (and hence to (NGP)) we only need the constant $V * 1$ be positive. If we take V as the potential given in the examples (i) or (ii), then $V \in L^1(\mathbb{R}^N)$ and

$$V * 1 = \int_{\mathbb{R}^N} V(x)\, dx > 0.$$

Therefore Theorem 3.1.2 also provides the global well-posedness for the equation (3.1.1). If we want to consider V as in the example (iii), the meaning of $\mathcal{K} * 1$ is not obvious. However, (3.1.16) still makes sense if $f \equiv 1$. In fact, using (3.1.16),

$$(\mathcal{K} * 1)(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\mathbb{S}^2} \frac{\Omega(\sigma)}{r^3} r^2\, d\sigma\, dr = 0.$$

Then if V is given by (3.1.17), $V * 1 = \alpha_1$ and we have the same conclusion as before, provided that $\alpha_1 > 0$.

One of the first works that introduces the nonlocal interaction in the Gross-Pitaevskii equation was made by Pomeau and Rica in [87] considering the potential (3.1.6). Their main purpose was to establish a model for superfluids with rotons. In fact, the Landau theory of superfluidity of Helium II says that the dispersion curve must exhibit a roton minimum (see [71, 37]) as was corroborated later by experimental observations ([34]). Although the model considered in [87] has a good fit with the roton minimum, it does not provide a correct sound speed. For this reason Berloff in [5] proposes the potential

$$W(x) = (\alpha + \beta A^2|x|^2 + \gamma A^4|x|^4)\exp(-A^2|x|^2), \quad x \in \mathbb{R}^3, \tag{3.1.19}$$

where the parameters A, α, β and γ are chosen such that the above requirements are satisfied. However, the existence of this roton minimum implies that $\widehat{W}$ must be negative in some interval. In addition, a numerical simulation in [5] shows that in this case the solution exhibits nonphysical mass concentration phenomenon, for certain initial conditions in $\phi + H^1(\mathbb{R}^3)$. At some point, our results are in agreement with these observations in the sense that Theorem 3.1.2 cannot be applied to the potential (3.1.19), because $\widehat{W}$ and W are negative in some interval. However, by Proposition 3.1.3 we may use the following local well-posedness result

Theorem 3.1.10. *Let W be a distribution satisfying $(\mathcal{W}_N)$. Then the Cauchy problem* (NGP) *is locally well-posed in $\phi + H^1(\mathbb{R}^N)$. More precisely, for every $w_0 \in H^1(\mathbb{R}^N)$ there exists $T > 0$ such that there is a unique $w \in C([-T,T], H^1(\mathbb{R}^N))$, for which $\phi + w$ solves* (NGP) *with the initial condition $u_0 = \phi + w_0$. In addition, w is defined on a maximal time interval $(-T_{\min}, T_{\max})$ where $w \in C^1((-T_{\min}, T_{\max}), H^{-1}(\mathbb{R}^N))$ and the blow-up alternative holds: $\|w(t)\|_{H^1(\mathbb{R}^N)} \to \infty$, as $t \to T_{\max}$ if $T_{\max} < \infty$ and $\|w(t)\|_{H^1(\mathbb{R}^N)} \to \infty$, as $t \to T_{\min}$ if $T_{\min} < \infty$. Furthermore, supposing that W is a real-valued even distribution, for any bounded closed interval $I \subset (-T_{\min}, T_{\max})$ the flow map $w_0 \in H^1(\mathbb{R}^N) \mapsto w \in C(I, H^1(\mathbb{R}^N))$ is continuous and the energy and the generalized momentum are conserved on $(-T_{\min}, T_{\max})$.*

It is an open question to establish which are the exact implications of change of sign of the Fourier transform of the potential for the global existence of the solutions of (NGP). As proposed in [6], a way to handle this problem would be to add a higher-order nonlinear term in (3.1.1) to avoid the mass concentration phenomenon, maintaining the correct phonon-roton dispersion curve.

This paper is organized as follows. In the next section we give several results about positive definite and positive distributions. In Section 3 we establish some convolution inequalities that involve the hypothesis $(\mathcal{W}_N)$ and we give the proof of Corollary 3.1.5. We prove the local well-posedness in Section 3.4 and also Propositions 3.1.4 and 3.1.7. Theorem 3.1.2 is completed in Section 3.5. In Section 3.6 we briefly recall the arguments that lead to Theorem 3.1.6 and in Section 3.7 we study the conservation of momentum and mass.

3.2 Positive definite and positive distributions

The purpose of this section is to recall some classical results for positive definite and positive distributions, in the context of Theorem 3.1.2. We also state some properties that we do not use in the next sections, but are useful to better understand the type of potentials considered in Theorem 3.1.2.

L. Schwartz in [95] defines that a (complex-valued) distribution T is positive definite if

$$\langle T, \phi * \breve{\phi} \rangle \geq 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N; \mathbb{C}), \tag{3.2.1}$$

with $\breve{\phi}(x) = \overline{\phi}(-x)$. In virtue of our hypothesis on W, we have preferred to adopt the simpler definition (3.1.9). The relation between these two possible definitions is given in the following lemma.

Lemma 3.2.1. *Let T be a real-valued distribution.*

(i) If T is positive definite (in the sense of (3.1.9)*) and even, then T fulfils* (3.2.1)*.*

(ii) If T verifies (3.2.1)*, then T is even.*

In particular, an even real-valued distribution is positive definite (in the sense of (3.1.9)*) if and only if it satisfies* (3.2.1)*.*

Proof. Suppose that T is positive definite in the sense of (3.1.9). Let $\phi \in C_0^\infty(\mathbb{R}^N; \mathbb{C})$, with $\phi = \phi_1 + i\phi_2$, $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^N; \mathbb{R})$. Then

$$\langle T, \phi * \breve{\phi} \rangle = \langle T, \phi_1 * \widetilde{\phi}_1 \rangle + \langle T, \widetilde{\phi}_2 * \phi_2 \rangle + i\langle T, \widetilde{\phi}_1 * \phi_2 \rangle - i\langle T, \phi_1 * \widetilde{\phi}_2 \rangle. \tag{3.2.2}$$

Since T is even,

$$\langle T, \widetilde{\phi}_1 * \phi_2 \rangle = \langle T, \phi_1 * \widetilde{\phi}_2 \rangle.$$

Therefore the imaginary part in the r.h.s. of (3.2.2) is zero. The real part is positive because T is positive definite, which implies that T verifies (3.2.1).

For the proof of (ii), see [95]. □

The next result characterizes the positive definite distributions under the hypotheses of Theorem 3.1.2. In particular, it gives a simple way to check the positive definiteness in terms of the Fourier transform.

Proposition 3.2.2. *Let $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ be an even real-valued distribution. The following assertions are equivalent*

(i) W is a positive definite distribution.

(*ii*) $\widehat{W} \in L^\infty(\mathbb{R}^N)$ *and* $\widehat{W}(\xi) \geq 0$ *for almost every* $\xi \in \mathbb{R}^N$.

(*iii*) *For every* $f \in L^2(\mathbb{R}^N;\mathbb{R})$,

$$\int_{\mathbb{R}^N} (W * f)(x) f(x)\, dx \geq 0.$$

Proof. (i) ⇒ (ii). By Lemma 3.2.1, we may apply the so-called Schwartz-Bochner Theorem (see [95], p. 276). Then there exists a positive measure $\mu \in S'(\mathbb{R}^N)$ such that $\widehat{W} = \mu$. Since $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$, we have that $\widehat{W} \in L^\infty(\mathbb{R}^N)$, and therefore $\widehat{W}$ is a nonnegative bounded function.

(ii) ⇒ (iii). Since $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$, $W * f \in L^2(\mathbb{R}^N)$. From the fact that $S(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, we also have that

$$\widehat{W * f} = \widehat{W}\widehat{f}.$$

Using that f is real-valued, by Parseval's theorem we finally deduce

$$\int_{\mathbb{R}^N} (W * f)(x) f(x)\, dx = (2\pi)^{-N} \int_{\mathbb{R}^N} \widehat{W}(\xi) |\widehat{f}(\xi)|^2 \, d\xi \geq 0,$$

where we have used that $\widehat{W} \geq 0$ for the last inequality.

(iii) ⇒ (i). This implication directly follows from the fact that $C_0^\infty(\mathbb{R}^N;\mathbb{R}) \subset L^2(\mathbb{R}^N;\mathbb{R})$. □

We remark that a positive definite distribution is not necessarily a positive distribution. For instance, we consider the Laguerre-Gaussian functions

$$W_m(x) = e^{-|x|^2} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \binom{m+\frac{N}{2}}{m-k} |x|^{2k}, \quad x \in \mathbb{R}^N,\ m \in \mathbb{N}. \tag{3.2.3}$$

These functions are negative in some subset of $\mathbb{R}^N$ and since $\widehat{W_m} \geq 0$ (see e.g. [36], p. 38), Proposition 3.2.2 shows that they are positive definite functions. We also have that $W_m \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then Corollary 3.1.5 gives global existence of (NGP) for the potential (3.2.3) in any dimension $N \geq 2$.

In the case that the considered distribution is actually a bounded function, its positive definiteness gives some regularity. In other direction, the concept of positive definiteness may be related to the same concept used for matrices. We recall some of these results in the next lemma.

Lemma 3.2.3. *Let W be an even real-valued positive definite distribution.*

(*i*) *If $W \in L^\infty(\mathbb{R}^N)$, then it coincides almost everywhere with a continuous function.*

(*ii*) *If W is continuous, then $W(0) = \|W\|_{L^\infty(\mathbb{R}^N)}$ and for all $x_1, \ldots, x_m \in \mathbb{R}^N$, $m \geq 1$, the matrix given by $A_{jk} = W(x_j - x_k)$, $j,k \in \{1,\ldots,m\}$, is a positive semi-definite matrix.*

Proof. Taking into consideration Lemma 3.2.1, these statements are proved in [95]. □

The importance of the condition (3.1.12) is that it gives the following coercivity property to the potential energy.

Lemma 3.2.4. *Assume that $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ verifies* (3.1.12). *Then for all $f \in L^2(\mathbb{R}^N;\mathbb{R})$,*

$$\sigma \|f\|_{L^2}^2 \leq \int_{\mathbb{R}^N} (W * f)(x) f(x)\, dx \leq \|W\|_{2,2} \|f\|_{L^2}^2. \tag{3.2.4}$$

Proof. The first inequality follows from Parseval's theorem,

$$\int_{\mathbb{R}^N} (W * f)(x) f(x)\, dx = (2\pi)^{-N} \int_{\mathbb{R}^N} \widehat{W}(\xi) |\widehat{f}(\xi)|^2 \, d\xi \geq \sigma \|f\|_{L^2}^2.$$

The second inequality in (3.2.4) is immediate since $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$. □

The purpose of the last lemma in this section is to establish some properties of the positive distributions which appear in Theorem 3.1.2. In particular, we show that for these distributions $(\mathcal{W}_N)$ is automatically verified if $1 \leq N \leq 3$.

Lemma 3.2.5. *Let $W \in \mathcal{M}_{1,1}(\mathbb{R}^N)$ be a positive distribution. Then $W \in \mathcal{M}_{p,p}(\mathbb{R}^N)$, for any $1 \leq p \leq \infty$ and W is a positive Borel measure of finite mass. If $1 \leq N \leq 3$ we also have that W satisfies $(\mathcal{W}_N)$.*

Proof. Since $W \in \mathcal{M}_{1,1}(\mathbb{R}^N)$, it is well known that W is a (complex-valued) finite Borel measure. Then $W \in \mathcal{M}_{\infty,\infty}(\mathbb{R}^N)$ and by interpolation $W \in \mathcal{M}_{p,p}(\mathbb{R}^N)$ for any $1 \leq p \leq \infty$. Finally, the fact that W is a positive distribution implies that it is a positive measure (cf. [95]). By Proposition 3.1.3 we conclude that W satisfies $(\mathcal{W}_N)$, if $1 \leq N \leq 3$. □

3.3 Some consequences of assumption $(\mathcal{W}_N)$

We first establish some inequalities involving the convolution with W that explain in part how the hypothesis $(\mathcal{W}_N)$ works. After that, we give the proof of Proposition 3.1.3 and Corollary 3.1.5.

From now on we adopt the standard notation $C(\cdot, \cdot, \dots)$ to represent a generic constant that depends only on each of its arguments, and possibly on some fixed numbers such as the dimension. In the case that $W \in \mathcal{M}_{p,q}(\mathbb{R}^N)$ we use $C(W)$ to denote a constant that only depends on the norm $\|W\|_{p,q}$. We also use the notation p' for the conjugate exponent of p given by $1/p + 1/p' = 1$.

Lemma 3.3.1. *Let $W \in \mathcal{M}_{p_1,q_1}(\mathbb{R}^N) \cap \mathcal{M}_{p_2,q_2}(\mathbb{R}^N) \cap \mathcal{M}_{p_3,q_3}(\mathbb{R}^N)$, with*

$$p_1, p_2, p_3, q_1, q_2, q_3 \geq 1 \quad \text{and} \quad \frac{1}{p_3} + \frac{1}{q_2} = \frac{1}{q_1}.$$

Suppose that there are $s_1, s_2 \geq 1$, such that

$$\frac{1}{p_1} - \frac{1}{p_3} = \frac{1}{s_1}, \quad \frac{1}{q_1} - \frac{1}{q_3} = \frac{1}{s_2}.$$

Then for any $u, v \in S(\mathbb{R}^N)$

$$\begin{aligned} \|(W * u)v\|_{L^{q_1}} &\leq \|W\|_{p_2,q_2} \|u\|_{L^{p_2}} \|v\|_{L^{p_3}}, \\ \|(W * u)v\|_{L^{q_1}} &\leq \|W\|_{p_3,q_3} \|u\|_{L^{p_3}} \|v\|_{L^{s_2}}, \\ \|W * (uv)\|_{L^{q_1}} &\leq \|W\|_{p_1,q_1} \|u\|_{L^{p_3}} \|v\|_{L^{s_1}}. \end{aligned}$$

Proof. The proof is a direct consequence of Hölder inequality and the hypotheses on W. □

Lemma 3.3.2. *Assume that W satisfies $(\mathcal{W}_N)$ and that $N \geq 4$. Then $W \in \mathcal{M}_{\frac{N}{N-2},2}(\mathbb{R}^N)$, $W \in \mathcal{M}_{\frac{N}{N-2},\frac{N}{2}}(\mathbb{R}^N)$ and $W \in \mathcal{M}_{2,\frac{N}{2}}(\mathbb{R}^N)$.*

Proof. From the Riesz-Thorin interpolation theorem and the fact that $\left(\frac{1}{2}, \frac{2}{N}\right)$ and $\left(\frac{N-2}{N}, \frac{2}{N}\right)$ belong to the convex hull of

$$\left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{p_1}, \frac{1}{q_1}\right), \left(\frac{1}{p_3}, \frac{1}{q_3}\right), \left(\frac{1}{p_4}, \frac{1}{q_4}\right)\right\},$$

we conclude that $W \in \mathcal{M}_{2,\frac{N}{2}}(\mathbb{R}^N)$ and $W \in \mathcal{M}_{\frac{N}{N-2},\frac{N}{2}}(\mathbb{R}^N)$. Since the conjugate exponent of $\frac{N}{N-2}$ is $\frac{N}{2}$, $W \in \mathcal{M}_{2,\frac{N}{2}}(\mathbb{R}^N)$ implies that $W \in \mathcal{M}_{\frac{N}{N-2},2}(\mathbb{R}^N)$. □

Lemma 3.3.3. *Assume that W satisfies ($\mathcal{W}_N$). Then for any $u, v, w \in S(\mathbb{R}^N)$,*

$$\|(W * (uv))w\|_{L^{\tilde{\gamma}}} \leq C(W)\|u\|_{L^{\tilde{s}}}\|v\|_{L^{\tilde{r}}}\|w\|_{L^{\tilde{r}}}, \tag{3.3.1}$$

for some $2 > \tilde{\gamma} > \frac{2N}{N+2}$, $\frac{2N}{N-2} > \tilde{r}, \tilde{s} > 2$ if $N \geq 3$, and $2 > \tilde{\gamma} > 1$, $\infty > \tilde{r}, \tilde{s} > 2$ if $N = 1, 2$.

Proof. If $N \geq 4$, by Lemma 3.3.2 we have that $W \in \mathcal{M}_{\frac{N}{N-2},\frac{N}{2}}(\mathbb{R}^N)$. Since also $W \in \mathcal{M}_{p_4,q_4}(\mathbb{R}^N)$, from the Riesz-Thorin interpolation theorem we deduce that there exist $\bar{p}$ and $\bar{q}$ such that

$$W \in \mathcal{M}_{\bar{p},\bar{q}}(\mathbb{R}^N), \quad \frac{N}{N-1} < \bar{p} < \frac{N}{N-2}, \quad \frac{N}{2} < \bar{q} < N. \tag{3.3.2}$$

Now we set

$$\frac{1}{\tilde{r}} = \min\left\{\frac{1}{2}\left(1 - \frac{1}{\bar{q}}\right), \frac{1}{2\bar{p}}\right\}, \ \frac{1}{\tilde{\gamma}} = \frac{1}{\bar{q}} + \frac{1}{\tilde{r}}, \ \frac{1}{\tilde{s}} = \frac{1}{\bar{p}} - \frac{1}{\tilde{r}}.$$

In view of (3.3.2), we have $\frac{2N}{N+2} < \tilde{\gamma} < 2$ and $2 < \tilde{r}, \tilde{s} < \frac{N-2}{2N}$. By Hölder inequality, we conclude that

$$\begin{aligned}\|(W * (uv))w\|_{L^{\tilde{\gamma}}} &\leq \|W * (uv)\|_{L^{\bar{q}}}\|w\|_{L^{\tilde{r}}} \\ &\leq \|W\|_{\bar{p},\bar{q}}\|uv\|_{L^{\bar{p}}}\|w\|_{L^{\tilde{r}}} \\ &\leq \|W\|_{\bar{p},\bar{q}}\|u\|_{L^{\tilde{s}}}\|v\|_{L^{\tilde{r}}}\|w\|_{L^{\tilde{r}}}.\end{aligned}$$

If $N = 1, 2, 3$, the proof is simpler. It is sufficient to take $\bar{q} = 2$, $\bar{p} = 2$, $\tilde{s} = \tilde{r} = 4$, $\tilde{\gamma} = \frac{4}{3}$ in the last inequality to deduce (3.3.1). □

Lemma 3.3.4. *Assume that W satisfies ($\mathcal{W}_N$).*

(i) For any $u \in \phi + H^1(\mathbb{R}^N)$ we have $(W * (1 - |u|^2))(1 - |u|^2) \in L^1(\mathbb{R}^N)$

(ii) If W is also an even real-valued distribution, then for any $u \in \phi + H^1(\mathbb{R}^N)$ and $h \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (W * \langle\!\langle u, h\rangle\!\rangle)(1 - |u|^2)\, dx = \int_{\mathbb{R}^N} (W * (1 - |u|^2))\langle\!\langle u, h\rangle\!\rangle\, dx. \tag{3.3.3}$$

Proof. Let $u = \phi + w$, with $w \in H^1(\mathbb{R}^N)$. If $N \geq 4$, by (3.1.10) and the Sobolev embedding theorem, we deduce that

$$(1 - |\phi|^2 - 2\langle\!\langle \phi, w\rangle\!\rangle - |w|^2) \in L^2(\mathbb{R}^N) + L^{\frac{N}{N-2}}(\mathbb{R}^N).$$

By Lemma 3.3.2 we have that the map $h \mapsto W * h$ is continuous from $L^2(\mathbb{R}^N) + L^{\frac{N}{N-2}}(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ and since $\frac{N-2}{N} + \frac{2}{N} = 1$, by Hölder inequality we conclude that

$$(W * (1 - |\phi|^2 - 2\langle\langle \phi, w\rangle\rangle - |w|^2))(1 - |\phi|^2 - 2\langle\langle \phi, w\rangle\rangle - |w|^2) \in L^1(\mathbb{R}^N). \tag{3.3.4}$$

If $1 \leq N \leq 3$, (3.3.4) follows from the fact that $|w|^2 \in L^2(\mathbb{R}^N)$. This concludes the proof of (i).

A similar argument shows that $\|(W * \langle\langle u, h\rangle\rangle)(1 - |u|^2)\|_{L^1} < \infty$. Then using that W is even and Fubini's theorem we obtain (ii). $\square$

The previous lemmas will be useful in the next sections, in particular to prove the local well-posedness of (NGP). Now we give the proofs of Proposition 3.1.3 and Corollary 3.1.5, that involve some straightforward computations.

Proof of Proposition 3.1.3. For the first part of (i), we note that the hypothesis implies that $W \in \mathcal{M}_{p,p}(\mathbb{R}^N)$ for any $\frac{3}{2} \leq p \leq 3$. Then it is sufficient to take $p_1 = q_1 = \frac{3}{2}$, $p_2 = p_3 = q_2 = q_3 = 3$ and $p_4 = q_4 = 2$ to see that $(\mathcal{W}_N)$ is fulfilled. For the second part of (i), we need prove that $W \in \mathcal{M}_{3,3}(\mathbb{R}^N)$. Recalling that $\mathcal{M}_{p,q}(\mathbb{R}^N) = \mathcal{M}_{q',p'}(\mathbb{R}^N)$ for $1 < p \leq q < \infty$ and using the Riesz interpolation theorem, we have that $W \in \mathcal{M}_{s,t}(\mathbb{R}^N)$, for every (s^{-1}, t^{-1}) in the convex hull of

$$\left\{ \left(\frac{1}{2}, \frac{1}{2} \right) \right\} \cup \bigcup_{j=1}^{3} \left\{ \left(\frac{1}{p_j}, \frac{1}{q_j} \right), \left(1 - \frac{1}{q_j}, 1 - \frac{1}{p_j} \right) \right\}. \tag{3.3.5}$$

By hypothesis, $p_i = q_i$, $i = 1, 2, 3$, thus $(\mathcal{W}_N)$ implies that

$$\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_1}, \quad 2 \geq p_1 \text{ and } \ p_2, p_3 \geq 2.$$

Hence the convex hull of (3.3.5) simplifies to

$$\left\{ (x, x) \in \mathbb{R}^2 : \min\left\{ 1 - \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{p_2} \right\} \leq x \leq \max\left\{ \frac{1}{p_1}, 1 - \frac{1}{p_2}, 1 - \frac{1}{p_1} + \frac{1}{p_2} \right\} \right\}.$$

Arguing by contradiction, it is simple to see that

$$\min\left\{ 1 - \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{p_2} \right\} \leq \frac{1}{3} \quad \text{and} \quad \frac{2}{3} \leq \max\left\{ \frac{1}{p_1}, 1 - \frac{1}{p_2}, 1 - \frac{1}{p_1} + \frac{1}{p_2} \right\}.$$

Therefore $W \in \mathcal{M}_{s,s}(\mathbb{R}^N)$, for every $\frac{3}{2} \leq s \leq 3$. In particular $W \in \mathcal{M}_{2,2}(\mathbb{R}^N) \cap \mathcal{M}_{3,3}(\mathbb{R}^N)$.

To prove (ii), we notice that by interpolation we have that $W \in \mathcal{M}_{\alpha,\beta}(\mathbb{R}^N)$, for all α, β satisfying

$$1 \leq \alpha, \beta, \quad \frac{1}{\alpha} - \left(1 - \frac{1}{\bar{r}} \right) \leq \frac{1}{\beta} \leq \frac{1}{\alpha}. \tag{3.3.6}$$

We now define

$$p_2 = p_3 = \begin{cases} 3, & \text{if } 4 \leq N \leq 5, \\ \frac{s_N}{s_N - 1}, & \text{if } 6 \leq N, \end{cases} \qquad q_2 = q_3 = \begin{cases} 3, & \text{if } 4 \leq N \leq 5, \\ N, & \text{if } 6 \leq N, \end{cases}$$

$$p_1 = \begin{cases} \frac{3}{2}, & \text{if } 4 \leq N \leq 5, \\ \frac{N}{N-1}, & \text{if } 6 \leq N, \end{cases} \qquad q_1 = \begin{cases} \frac{3}{2}, & \text{if } 4 \leq N \leq 5, \\ \frac{p_3 q_2}{p_3 + q_2}, & \text{if } 6 \leq N, \end{cases}$$

$p_4 = \frac{2\bar{r}}{2\bar{r}-1}$, $q_4 = 2\bar{r}$, where

$$s_N = \begin{cases} \dfrac{N}{4} + \varepsilon_N, & \text{if } 6 \le N \le 7, \\ \dfrac{2(N+1)}{N+2}, & \text{if } 8 \le N, \end{cases}$$

and $\varepsilon_N > 0$ is chosen small enough such that $0 < \varepsilon_N < 2 - \frac{N}{4}$ if $6 \le N \le 7$. Then we have that

$$\frac{2N}{N+2} < s_N < 2, \qquad \text{for any } N \ge 6. \tag{3.3.7}$$

Using that $\bar{r} > \frac{N}{4}$ and (3.3.7), we can verify that the choice of (p_i, q_i), $i \in \{1, \dots, 4\}$, satisfies (3.3.6) with $\alpha = p_i$ and $\beta = q_i$, as well as all the others restrictions in the hypothesis $(\mathcal{W}_N)$, which completes the proof. □

Proof of Corollary 3.1.5. By Young inequality we have that $W \in \mathcal{M}_{p,p}(\mathbb{R}^N)$, for any $1 \le p \le \infty$. In particular the condition $W \in \mathcal{M}_{1,1}(\mathbb{R}^N)$ is fulfilled. If $1 \le N \le 3$, the conclusion is a consequence of Proposition 3.1.3 and Theorem 3.1.2. If $N \ge 4$, by Young inequality we have that $W \in \mathcal{M}_{p,q}(\mathbb{R}^N)$, for all $1 - \frac{1}{r} \le \frac{1}{p} \le 1$, with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$. Then the proof follows again from Proposition 3.1.3 and Theorem 3.1.2. □

3.4 Local existence

In order to prove Theorem 3.1.2 we first are going to prove the local well-posedness. Theorem 3.1.10 is based on the fact that if we set $u = w + \phi$, then u is a solution of (NGP) with initial condition $u_0 = \phi + w_0$ if and only if w solves

$$\begin{cases} i\partial_t w + \Delta w + f(w) = 0 \text{ on } \mathbb{R}^N \times \mathbb{R}, \\ w(0) = w_0, \end{cases} \tag{3.4.1}$$

with

$$f(w) = \Delta\phi + (w + \phi)(W * (1 - |\phi + w|^2)).$$

We decompose f as

$$f(w) = g_1(w) + g_2(w) + g_3(w) + g_4(w), \tag{3.4.2}$$

with

$$\begin{aligned} g_1(w) &= \Delta\phi + (W * (1 - |\phi|^2))\phi, \\ g_2(w) &= -2(W * \langle\!\langle \phi, w \rangle\!\rangle)\phi, \\ g_3(w) &= -(W * |w|^2)\phi - 2(W * \langle\!\langle \phi, w \rangle\!\rangle)w + (W * (1 - |\phi|^2))w, \\ g_4(w) &= -(W * |w|^2)w. \end{aligned}$$

The next lemma gives some estimates on each of these functions.

Lemma 3.4.1. *Assume that W satisfies $(\mathcal{W}_N)$. Using the numbers given by $(\mathcal{W}_N)$ and Lemma 3.3.3, let $r_1 = r_2 = 2$, $r_3 = p_3$, $r_4 = \tilde{r}$, $\rho_1 = \rho_2 = 2$, $\rho_3 = q_1'$ and $\rho_4 = \tilde{\gamma}'$. Then*

$$g_j \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N)), \ j \in \{1, 2, 3, 4\}. \tag{3.4.3}$$

Furthermore, for any $M > 0$ there exists a constant $C(M, W, \phi)$ such that

$$\|g_j(w_1) - g_j(w_2)\|_{L^{\rho_j'}} \leq C(M, W, \phi)\|w_1 - w_2\|_{L^{r_j}}, \tag{3.4.4}$$

for all $w_1, w_2 \in H^1(\mathbb{R}^N)$ with $\|w_1\|_{H^1}, \|w_2\|_{H^1} \leq M$, and

$$\|g_j(w)\|_{W^{1,\rho_j'}} \leq C(M, W, \phi)(1 + \|w\|_{W^{1,r_j}}), \tag{3.4.5}$$

for all $w \in H^1(\mathbb{R}^N) \cap W^{1,r_j}(\mathbb{R}^N)$ with $\|w\|_{H^1} \leq M$.

Proof. Since g_1 is a constant function of w, $g_1 \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$ and (3.4.4) is trivial in this case. The condition (3.4.5) follows from the estimate

$$\|g_1(w)\|_{H^1} \leq \|\nabla\phi\|_{H^2} + \|W\|_{2,2}(\|1 - |\phi|^2\|_{L^2}\|\phi\|_{W^{1,\infty}} + 2\|\phi\|^2_{L^\infty}\|\nabla\phi\|_{L^2}).$$

Similarly we obtain for g_2,

$$\|g_2(w_1) - g_2(w_2)\|_{L^2} \leq 2\|W\|_{2,2}\|\phi\|^2_{L^\infty}\|w_1 - w_2\|_{L^2}$$

and

$$\begin{aligned}\|\nabla g_2(w)\|_{L^2} &\leq 2\|W\|_{2,2}\|\phi\|_{L^\infty}\big(\|\phi\|_{L^\infty}\|\nabla w\|_{L^2} + 2\|\nabla\phi\|_{L^\infty}\|w\|_{L^2}\big)\\ &\leq C(W, \phi)\|w\|_{H^1}.\end{aligned}$$

Then we deduce (3.4.4) and (3.4.5) for $j = 2$.

For g_3, we have

$$\begin{aligned}g_3(w_2) - g_3(w_1) = (W * (|w_1|^2 - |w_2|^2))\phi + 2(W * \langle\!\langle \phi, w_1 - w_2\rangle\!\rangle)w_1&\\ + 2(W * \langle\!\langle \phi, w_2\rangle\!\rangle)(w_1 - w_2) + (W * (1 - |\phi|^2))(w_1 - w_2).&\end{aligned}$$

The assumption $(\mathcal{W}_N)$ allows to apply Lemma 3.3.1 and then we derive

$$\begin{aligned}\|g_3(w_2) - g_3(w_1)\|_{L^{\rho_3'}} \leq C(W, \phi)\|w_1 - w_2\|_{L^{r_3}}(\|w_1\|_{L^{s_1}} + \|w_2\|_{L^{s_1}}&\\ +2\|w_1\|_{L^{s_2}} + 2\|w_2\|_{L^{p_2}} + 1).&\end{aligned} \tag{3.4.6}$$

More precisely, the dependence on ϕ of the constant $C(W, \phi)$ in the last inequality is given explicitly by $\max\{\|\phi\|_{L^\infty}, \|1 - |\phi|^2\|_{L^{p_2}}\}$. By the Sobolev embedding theorem

$$H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad \forall\, p \in \left[2, \frac{2N}{N-2}\right] \text{ if } N \geq 3 \text{ and } \forall\, p \in [2, \infty) \text{ if } N = 1, 2. \tag{3.4.7}$$

In particular,

$$\|w_1\|_{L^{s_1}} + \|w_2\|_{L^{s_1}} + 2\|w_1\|_{L^{s_2}} + 2\|w_2\|_{L^{p_2}} \leq C(\|w_1\|_{H^1} + \|w_2\|_{H^1}),$$

which together with (3.4.6) gives us (3.4.4) for g_3. With the same type of computations, taking $w \in H^1(\mathbb{R}^N)$, $\|w\|_{H^1} \leq M$, we have

$$\|\nabla g_3(w)\|_{L^{\rho_3'}} \leq C(M, W, \phi)(\|\nabla w\|_{L^{r_3}} + \|w\|_{L^{r_3}}),$$

where the dependence on ϕ is in terms of $\|\phi\|_{L^\infty}$, $\|\nabla\phi\|_{L^\infty}$, $\|1 - |\phi|^2\|_{L^{p_2}}$ and $\|\nabla\phi\|_{L^{p_2}}$.

For g_4, applying Lemma 3.3.3 we obtain

$$\|g_4(w_1)-g_4(w_2)\|_{L^{\rho_4'}} \leq C(W)\|w_1-w_2\|_{L^{r_4}}((\|w_1\|_{L^s}+\|w_2\|_{L^s})\|w_1\|_{L^{r_4}} + \|w_2\|_{L^s}\|w_2\|_{L^{r_4}})$$

and

$$\|\nabla g_4(w)\|_{L^{\rho_4'}} \leq C(W)\|\nabla w\|_{L^{r_4}}\|w\|_{L^{r_4}}\|w\|_{L^s}.$$

As before, using (3.4.7), we conclude that g_4 verifies (3.4.4)-(3.4.5).

Since for $2 \leq j \leq 4$, $2 \leq r_j < \frac{2N}{N-2}$ ($2 \leq r_j < \infty$ if $N = 1, 2$), we have the continuous embeddings

$$H^1(\mathbb{R}^N) \hookrightarrow L^{r_j}(\mathbb{R}^N) \quad \text{and} \quad L^{r_j'}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N).$$

Then inequality (3.4.4) implies (3.4.3), for $j \in \{2,3,4\}$. □

Now we analyze the potential energy associated to (3.4.1). For any $v \in H^1(\mathbb{R}^N)$ we set

$$F(v) := \int_{\mathbb{R}^N} \langle\!\langle \Delta\phi, v \rangle\!\rangle \, dx - \frac{1}{4}\int_{\mathbb{R}^N} (W * (1-|\phi+v|^2))(1-|\phi+v|^2)\, dx, \tag{3.4.8}$$

and using the notation of Lemma 3.4.1, we fix for the rest of this section

$$r = \max\{r_1, r_2, r_3, r_4, \rho_1, \rho_2, \rho_3, \rho_4\}. \tag{3.4.9}$$

Lemma 3.4.2. *Assume that W satisfies $(\mathcal{W}_N)$. Then the functional F is well-defined on $H^1(\mathbb{R}^N)$. If moreover W is a real-valued even distribution, we have the following properties.*

(i) F is Fréchet-differentiable and

$$F \in C^1(H^1(\mathbb{R}^N), \mathbb{R}) \text{ with } F' = f. \tag{3.4.10}$$

(ii) For any $M > 0$, there exists a constant $C(M, W, \phi)$ such that

$$|F(u)-F(v)| \leq C(M,W,\phi)(\|u-v\|_{L^2} + \|u-v\|_{L^r}), \tag{3.4.11}$$

for any $u, v \in H^1(\mathbb{R}^N)$, with $\|u\|_{H^1}, \|v\|_{H^1} \leq M$.

Proof. By Lemma 3.3.4, F is well-defined in $H^1(\mathbb{R}^N)$ for any N. To prove (i), we compute now the Gâteaux derivative of F. For $h \in H^1(\mathbb{R}^N)$ we have

$$\begin{aligned} d_G F(v)[h] &= \lim_{t\to 0} \frac{F(v+th)-F(v)}{t} \\ &= \int_{\mathbb{R}^N} \langle\!\langle \Delta\phi, h \rangle\!\rangle \, dx + \frac{1}{2}\int_{\mathbb{R}^N} (W * \langle\!\langle \phi+v, h \rangle\!\rangle)(1-|\phi+v|^2)\, dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N} (W*(1-|\phi+v|^2))\langle\!\langle \phi+v, h\rangle\!\rangle \, dx. \end{aligned}$$

Since W is an even distribution, (3.3.3) implies that the last two integrals are equal. Finally we get that

$$d_G F(v)[h] = \int_{\mathbb{R}^N} \langle\!\langle f(v), h \rangle\!\rangle \, dx = \langle f(v), h \rangle_{H^{-1}, H^1}.$$

From (3.4.2) and (3.4.3), we have that $f \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$. Hence the map $v \to d_G F(v)$ is continuous from $H^1(\mathbb{R}^N)$ to $H^{-1}(\mathbb{R}^N)$, which implies that F is continuously Fréchet-differentiable and satisfies (3.4.10).

For the proof of (ii), using (3.4.10) and the mean-value theorem, we have

$$F(u) - F(v) = \int_0^1 \frac{d}{ds} F(su + (1-s)v)\, ds = \int_0^1 \langle f(su + (1-s)v), u - v\rangle_{H^{-1},H^1}\, ds.$$

Then by Lemma 3.4.1,

$$\begin{aligned} |F(u) - F(v)| &\leq \sup_{s\in[0,1]} \sum_{j=1}^{4} \|g_j(su + (1-s)v)\|_{L^{\rho_j'}} \|u - v\|_{L^{\rho_j}} \\ &\leq \sum_{j=1}^{4} C(M, W, \phi)(\|u\|_{L^{r_j}} + \|v\|_{L^{r_j}} + 1)\|u - v\|_{L^{\rho_j}}. \end{aligned} \tag{3.4.12}$$

Since we assume that $\|u\|_{H^1}, \|v\|_{H^1} \leq M$, (3.4.7) implies that

$$\|u\|_{L^{r_j}} + \|v\|_{L^{r_j}} + 1 \leq C(M). \tag{3.4.13}$$

Also, it follows from L^p-interpolation and Young's inequality that

$$\|u - v\|_{L^{\rho_j}} \leq \|u - v\|_{L^2}^{\theta_j} \|u - v\|_{L^r}^{1-\theta_j} \leq \|u - v\|_{L^2} + \|u - v\|_{L^r}, \tag{3.4.14}$$

with $\theta_j = \frac{2(r-\rho_j)}{\rho_j(r-2)}$. By combining (3.4.12), (3.4.13) and (3.4.14), we obtain (ii). □

Proof of Theorem 3.1.10. Recalling that r was fixed in (3.4.9), we define q by $\frac{1}{q} = \frac{N}{2}\left(\frac{1}{2} - \frac{1}{r}\right)$. Given $T, M > 0$, we consider the complete metric space

$$\begin{aligned} X_{T,M} = \{ w \in L^\infty((-T,T), H^1(\mathbb{R}^N)) \cap L^q((-T,T), W^{1,r}(\mathbb{R}^N)) : \\ \|w\|_{L^\infty((-T,T),H^1)} \leq M,\ \|w\|_{L^q((-T,T),W^{1,r})} \leq M\}, \end{aligned}$$

endowed with the distance

$$d_T(w_1, w_2) = \|w_1 - w_2\|_{L^\infty((-T,T),L^2)} + \|w_1 - w_2\|_{L^q((-T,T),L^r)}. \tag{3.4.15}$$

The estimates given in Lemmas 3.4.1, 3.4.2 and the Strichartz estimates show that the functional

$$\Phi(w) = e^{it\Delta} w_0 + i \int_0^t e^{i(t-s)\Delta} f(w(s))\, ds$$

is a contraction in $X_{T,M}$ for some $M \leq C(\|w_0\|_{H^1} + 1)$ and T small enough, but depending only on $\|w_0\|_{H^1}$. Then we have a solution given by Banach's fixed-point theorem. The arguments to complete Theorem 3.1.10 are rather standard. For instance, Theorem 4.4.6 in [24] automatically implies the existence, uniqueness, the blow-up alternative and that the function $L(t)$ given by

$$L(t) := L_1(t) + \frac{1}{4} \int_{\mathbb{R}^N} (W * (1 - |\phi + w(t)|^2))(1 - |\phi + w(t)|^2)\, dx,$$

with

$$L_1(t) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w(t)|^2\, dx - \int_{\mathbb{R}^N} \langle\!\langle \Delta\phi, w(t) \rangle\!\rangle\, dx,$$

is constant for all $t \in (-T_{\min}, T_{\max})$. Noticing that

$$L_1(t) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w(t) + \nabla\phi|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2\, dx,$$

we conclude that the energy is conserved.

However, the continuous dependence on the initial data in $H^1(\mathbb{R}^N)$ is not obvious, because the distance (3.4.15) does not involve derivatives. Therefore we give the complete proof of this point. Here we will omit the dependence on W and ϕ in the generic constant C, since it plays no role in the analysis of continuous dependence. Let $w_{0,n}, w_0 \in H^1(\mathbb{R}^N)$ be such that

$$w_{0,n} \to w_0 \quad \text{in } H^1(\mathbb{R}^N).$$

Then for some $n_0 \geq 0$,

$$\|w_{0,n}\|_{H^1} \leq \|w_0\|_{H^1} + 1, \quad \forall n \geq n_0.$$

We denote w_n and w the solutions with initial data $w_{0,n}$ and w_0, respectively. Then by the fixed-point argument, there exist $T > 0$ and a constant $C(\|w_0\|_{H^1})$, both depending only on $\|w_0\|_{H^1}$, such that w_n and w are defined in $[-T, T]$ for all $n \geq n_0$ and

$$\|w_n\|_{L^\infty((-T,T),H^1)} + \|w\|_{L^\infty((-T,T),H^1)} \leq C(\|w_0\|_{H^1}), \quad \forall n \geq n_0. \tag{3.4.16}$$

Since

$$w_n(t) - w(t) = e^{it\Delta}(w_{0,n} - w_0) + i\int_0^t e^{i(t-s)\Delta}(f(w_n(s)) - f(w(s)))\, ds,$$

using Strichartz estimates we have that

$$d_T(w_n, w) \leq C\|w_{0,n} - w_0\|_{L^2} + C\sum_{j=1}^{4} \|g_j(w_n) - g_j(w)\|_{L^{\gamma_j'}((-T,T),L^{\rho_j'})}, \tag{3.4.17}$$

with $\frac{1}{\gamma_j} = \frac{N}{2}\left(\frac{1}{2} - \frac{1}{\rho_j}\right)$. By Lemma 3.4.1, (3.4.16), using as in (3.4.14) an L^p-interpolation inequality and Young's inequality, we deduce that

$$\|g_j(w_n) - g_j(w)\|_{\rho_j'} \leq C(\|w_0\|_{H^1})(\|w_n - w\|_{L^2} + \|w_n - w\|_{L^r}). \tag{3.4.18}$$

Applying Hölder inequality with $\beta_j = \frac{1}{\gamma_j'} - \frac{1}{q}$,

$$\|w_n - w\|_{L^{\gamma_j'}((-T,T),L^r)} \leq \|w_n - w\|_{L^q((-T,T),L^r)}(2T)^{\beta_j}. \tag{3.4.19}$$

Notice that $0 < \beta_j \leq 1$ since $2 \leq \rho_j, r_j < \frac{2N}{N-2}$. Assuming $T \leq 1$ and putting together (3.4.18) and (3.4.19) we conclude that

$$\|g_j(w_n) - g_j(w)\|_{L^{\gamma_j'}((-T,T),L^{\rho_j'})} \leq C(\|w_0\|_{H^1})T^\beta d_T(w_n, w), \tag{3.4.20}$$

with $\beta = \min\{\beta_j, 1/\gamma_j' :\ 1 \leq j \leq 4\}$. Choosing T such that $4T^\beta C(\|w_0\|_{H^1}) \leq \frac{1}{2}$, (3.4.17) and (3.4.20) give

$$d_T(w_n, w) \leq 2C(\|w_0\|_{H^1})\|w_{0,n} - w_0\|_{H^1}.$$

Hence

$$w_n \to w, \text{ in } C([-T,T], L^2(\mathbb{R}^N)) \cap L^q((-T,T), L^r(\mathbb{R}^N)).$$

Thus from (3.4.16) and the Gagliardo-Nirenberg inequality, we conclude that $w_n \to w$ in $C([-T,T], L^p(\mathbb{R}^N))$, for every $2 \leq p < \infty$ if $N = 1,2$ and $2 \leq p < \frac{2N}{N-2}$ if $N \geq 3$. Using the inequality (3.4.11) in Lemma 3.4.2, it follows that $F(w_n) \to F(w)$ in $C([-T,T])$. Since the energy is conserved for w and w_n, this implies that

$$\|\nabla w_n\|_{L^2} \to \|\nabla w\|_{L^2} \ \text{ in } \ C([-T,T]).$$

In addition, from the equation $i\partial_t w_n = -\Delta w_n - f(w_n)$ in $[-T,T]$, we get

$$\|\partial_t w_n\|_{H^{-1}} \leq \|w_n\|_{H^1} + \sum_{j=1}^{4} \|g_j(w_n)\|_{H^{-1}},$$

Hence Lemma 3.4.1 and (3.4.16) provide a uniform bound for w_n in $C^1([-T,T], H^{-1}(\mathbb{R}^N))$. Therefore $w_n \to w$ in $C([-T,T], H^1(\mathbb{R}^N))$ (see Proposition 1.3.14 in [24]). A covering argument allows us to finish the proof in any closed bounded interval.

Since the generalized momentum still needs a precise definition, we will postpone the proof of its conservation until Section 3.7. □

We prove now Propositions 3.1.4 and 3.1.7 because the arguments involved are very similar to those used in this section. For these proofs we suppose that Theorem 3.1.2 is already proved.

Proof of Proposition 3.1.4. Let $u_n = \phi + w_n$ and $u_\infty = \phi + w_\infty$, where $w_n, w_\infty \in C(\mathbb{R}, H^1(\mathbb{R}^N))$, be the global solution of (NGP) with potentials W_n and W_∞, respectively, with the same initial data $u_0 = \phi + w_0$, with $w_0 \in H^1(\mathbb{R}^N)$. In the same spirit of the proof of Theorem 3.1.10, for $v \in H^1(\mathbb{R}^N)$, we set

$$f_n(v) = g_{1,n}(v) + g_{2,n}(v) + g_{3,n}(v) + g_{4,n}(v),$$

with

$$\begin{aligned}
g_{1,n}(v) &= \Delta\phi + (W_n * (1 - |\phi|^2))\phi, \\
g_{2,n}(v) &= -2(W_n * \langle\!\langle \phi, v \rangle\!\rangle)\phi, \\
g_{3,n}(v) &= -(W_n * |v|^2)\phi - 2(W_n * \langle\!\langle \phi, v \rangle\!\rangle)w + (W_n * (1 - |\phi|^2))v, \\
g_{4,n}(v) &= -(W_n * |v|^2)v,
\end{aligned}$$

for any $n \in \mathbb{N} \cup \{\infty\}$. Noticing that for any $v_1, v_2 \in H^1(\mathbb{R}^N)$, $1 \leq j \leq 4$,

$$g_{j,n}(v_1) - g_{j,m}(v_2) = (g_{j,n}(v_1) - g_{j,n}(v_2)) + (g_{j,n}(v_2) - g_{j,m}(v_2)),$$

Proposition 3.1.3, Lemma 3.3.1, the proof of Lemma 3.3.3 and the same argument given in Lemma 3.4.1 allows us to conclude that (we omit from now on the dependence on ϕ)

$$\|g_{j,n}(v_1) - g_{j,m}(v_2)\|_{L^{\rho_j'}} \leq C(W_n, M)\|v_1 - v_2\|_{L^{r_j}} + C(W_n - W_m, M)(\|v_2\|_{L^{r_j}} + 1), \quad (3.4.21)$$

for any $n, m \in \mathbb{N} \cup \{\infty\}$ and $v_1, v_2 \in H^1(\mathbb{R}^N)$ with $\|v_1\|_{H^1}, \|v_2\|_{H^1} \leq M$, with (the new choice of) ρ_j, r_j given by

$$\rho_1 = \rho_2 = r_1 = r_2 = 2, \ \rho_3 = r_3 = 3, \ \rho_4 = r_4 = 4, \quad (3.4.22)$$

and

$$C(W, M) = \sigma(W)C(M), \ \text{with } \sigma(W) = \max\{\|W\|_{2,2}, \|W\|_{3,3}\}. \quad (3.4.23)$$

By the uniqueness provided by Theorem 3.1.2, the functions w_n are given by the fixed-point argument of the proof of Theorem 3.1.10. Since the estimates for the fixed point can be obtained using Lemma 3.4.1, but with the values in (3.4.22), and by (3.1.14) we may assume that for $k = 2, 3$

$$\frac{1}{2}\|W_\infty\|_{k,k} \le \|W_n\|_{k,k} \le 2\|W_\infty\|_{k,k},$$

so that we have uniform bounds on W_n. Therefore we conclude that there exist some $T \le 1$ and $C > 0$ that only depend on $\|w_0\|_{H^1}$, $\|W_\infty\|_{2,2}$ and $\|W_\infty\|_{3,3}$ such that

$$\|w_n\|_{L^\infty((-T,T),H^1)} \le C, \quad \text{for any } n \in \mathbb{N} \cup \{\infty\}. \tag{3.4.24}$$

Using the distance

$$d_T(w_1, w_2) = \|w_1 - w_2\|_{L^\infty((-T,T),L^2)} + \|w_1 - w_2\|_{L^{\frac{8}{N}}((-T,T),L^4)},$$

the estimates (3.4.21), (3.4.24) and following the lines of the proof of Theorem 3.1.10, it leads to

$$d_T(w_n, w_\infty) \le C\sigma(W_n - W_\infty).$$

Hence the hypothesis (3.1.14) and (3.4.23) imply that

$$w_n \to w_\infty \quad \text{in } C([-T,T], L^2(\mathbb{R}^N)) \cap L^{\frac{8}{N}}((-T,T), L^4(\mathbb{R}^N)).$$

Then (3.4.24) and the Gagliardo-Nirenberg inequality imply that

$$w_n \to w_\infty \ \text{ in } C([-T,T], L^p(\mathbb{R}^N)), \ \forall p \in [2,\infty) \text{ if } N = 1,2 \text{ and } \forall p \in \left[2, \frac{2N}{N-2}\right) \ \text{ if } N \ge 3. \tag{3.4.25}$$

We denote by F_n the function given by (3.4.8), with W replaced by W_n, so that the conserved energy for each u_n is

$$E_n(u_n(t)) = \|\nabla w_n(t)\|_{L^2} + F_n(w_n(t)) = \|\nabla w_0\|_{L^2} + F_n(w_0), \quad \text{for any } t \in \mathbb{R}. \tag{3.4.26}$$

The inequality (3.4.21) and similar arguments as in the proof of Lemma 3.4.2 give for any $v_1, v_2 \in H^1(\mathbb{R}^N)$ with $\|v_1\|_{H^1}, \|v_2\|_{H^1} \le M$, that there exists a constant C depending only on M, $\|W_\infty\|_{2,2}$ and $\|W_\infty\|_{3,3}$, such that

$$|F_n(v_1) - F_m(v_2)| \le C\left(\|v_1 - v_2\|_{L^2} + \|v_1 - v_2\|_{L^4}\right) + C\sigma(W_n - W_m). \tag{3.4.27}$$

By putting together (3.4.24), (3.4.25) and (3.4.27), we deduce that $F_n(w_n) \to F_\infty(w_\infty)$ in $C([-T,T])$. Then by (3.4.26) we have that $\|\nabla w_n\|_{L^2} \to \|\nabla w_\infty\|_{L^2}$ in $C([-T,T])$. The conclusion follows as in the proof of Theorem 3.1.10. □

Proof of Proposition 3.1.7. Using the notation introduced at the beginning of this section, by Lemma 5.3.1 in [24], we only need to prove that for any $1 \le j \le 4$ and any $w \in H^s(\mathbb{R}^N)$ such that $\|w\|_{H^1} \le M$, we have

$$\|g_j(w)\|_{L^2} \le C(W, M, \phi)\left(1 + \|w\|_{H^s}\right), \tag{3.4.28}$$

for some $0 < s < 2$. From the estimate (3.4.5) in Lemma 3.4.1 and the Sobolev embedding theorem, we have the inequality (3.4.28) for $j = 1, 2$ for any $s \geq 1$. For $j = 3, 4$ we note that by the Sobolev embedding theorem,

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N), \ \forall p \in \left[\frac{2N}{N+2}, 2\right] \text{ if } N \geq 3 \text{ and } \ \forall p \in [1, 2] \text{ if } N = 1, 2,$$

and for any

$$r \in \left[2, \frac{2N}{N-2}\right], \text{ if } N \geq 3 \ \text{ and } \ r \in [2, \infty) \text{ if } N = 1, 2,$$

there exists $\frac{3}{2} < s < 2$ such that $H^s(\mathbb{R}^N) \hookrightarrow W^{1,r}(\mathbb{R}^N)$. Thus we have for $j = 3, 4$ that $W^{1,\rho_j'}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ and $H^{s_j}(\mathbb{R}^N) \hookrightarrow W^{1,r_j}(\mathbb{R}^N)$, for some $s_j < 2$. Setting $s = \max\{s_3, s_4\}$, from the inequality (3.4.5) we obtain estimate (3.4.28) $\square$

3.5 Global existence

In order to complete the proof of Theorem 3.1.2 we need to prove that the solutions given by Theorem 3.1.10 are global. We do this by establishing an appropriate estimate for $\|w(t)\|_{H^1}$. We distinguish three subcases, associated to the different assumptions on W.

Proof of Theorem 3.1.2-(i)-(a). We recall that by Theorem 3.1.10 we already have the conservation of energy

$$E_0 = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w(t) + \nabla \phi|^2 \, dx + \frac{1}{4}\int_{\mathbb{R}^N} (W * (|\phi + w(t)|^2 - 1))(|\phi + w(t)|^2 - 1) \, dx, \qquad (3.5.1)$$

for any $t \in (-T_{\min}, T_{\max})$. Since we are assuming that W is a positive definite distribution, the potential energy, i.e. the second integral in (3.5.1), is nonnegative. Hence

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w(t) + \nabla \phi|^2 \, dx \leq E_0$$

and using the elementary inequality

$$\int_{\mathbb{R}^N} |\nabla w \nabla \phi| \, dx \leq \frac{1}{4}\|\nabla w\|_{L^2}^2 + \|\nabla \phi\|_{L^2}^2, \qquad (3.5.2)$$

we conclude that

$$\|\nabla w(t)\|_{L^2}^2 \leq 4E_0 + 2\|\nabla \phi\|_{L^2}^2, \quad t \in (T_{\min}, T_{\max}), \qquad (3.5.3)$$

which gives a uniform bound for $\|\nabla w(t)\|_{L^2}$. Therefore we only need an appropriate bound for $\|w(t)\|_{L^2}$ to conclude that

$$\sup\{\|w(t)\|_{H^1} : t \in (-T_{\min}, T_{\max})\} < \infty. \qquad (3.5.4)$$

In virtue of the blow-up alternative in Theorem 3.1.10, we will deduce from (3.5.4) that $T_{\max} = T_{\min} = \infty$, which will complete the proof.

Now we prove the bound for $\|w(t)\|_{L^2}$. For any $t \in (-T_{\min}, T_{\max})$, we multiply (in the $H^{-1} - H^1$ duality sense) the equation (3.4.1) by iw, to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 =& \operatorname{Re}\int_{\mathbb{R}^N} if(w(t))\overline{w}(t)\,dx\\ =& -\operatorname{Im}\int_{\mathbb{R}^N}(\Delta\phi + \phi(W*(1-|\phi+w(t)|^2))\overline{w}(t)\,dx.\end{aligned}$$

Then

$$\frac{1}{2}\left|\frac{d}{dt}\|w(t)\|_{L^2}^2\right| \leq \|\Delta\phi\|_{L^2}\|w(t)\|_{L^2} + \|\phi\|_{L^\infty}\int_{\mathbb{R}^N}|W*(|\phi+w(t)|^2-1)||w(t)|\,dx. \tag{3.5.5}$$

We bound the last integral in (3.5.5) by $H_1(t) + H_2(t)$, with

$$\begin{aligned}H_1(t) =& \int_{\mathbb{R}^N}|W*(|\phi|^2 - 1 + 2\langle\!\langle \phi, w(t)\rangle\!\rangle)||w(t)|\,dx,\\ H_2(t) =& \int_{\mathbb{R}^N}|W*|w(t)|^2||w(t)|\,dx.\end{aligned}$$

Since $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$,

$$\begin{aligned}|H_1(t)| \leq& \|W*(|\phi|^2-1+2\langle\!\langle\phi,w\rangle\!\rangle)\|_{L^2}\|w(t)\|_{L^2}\\ \leq& \|W\|_{2,2}\big(\||\phi|^2-1\|_{L^2} + 2\|\phi\|_{L^\infty}\|w(t)\|_{L^2}\big)\|w(t)\|_{L^2}.\end{aligned}$$

Therefore we have

$$|H_1(t)| \leq C(W,\phi)(1+\|w(t)\|_{L^2}^2). \tag{3.5.6}$$

If $N \geq 4$, by Lemma 3.3.2 and the Sobolev embedding theorem,

$$\begin{aligned}|H_2(t)| \leq& \|W*|w(t)|^2\|_{L^2}\|w(t)\|_{L^2}\\ \leq& C(W)\|w(t)\|^2_{L^{\frac{2N}{N-2}}}\|w(t)\|_{L^2}\\ \leq& C(W)\|\nabla w(t)\|_{L^2}^2\|w(t)\|_{L^2}.\end{aligned}$$

By (3.5.3) we conclude that

$$|H_2(t)| \leq C(W,\phi,E_0)\|w(t)\|_{L^2}, \quad \text{for all } N \geq 4. \tag{3.5.7}$$

If $N = 2,3$, we only need to use that $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$, together with the Gagliardo-Nirenberg inequality. In fact,

$$\begin{aligned}|H_2(t)| \leq& \|W*|w(t)|^2\|_{L^2}\|w(t)\|_{L^2}\\ \leq& C(W)\|w(t)\|_{L^4}^2\|w(t)\|_{L^2}\\ \leq& C(W)\|\nabla w(t)\|_{L^2}^{\frac{N}{2}}\|w(t)\|_{L^2}^{3-\frac{N}{2}}.\end{aligned}$$

Since we are considering $N = 2,3$, using (3.5.3) it follows that

$$\|H_2(t)\|_{L^2} \leq C(W,\phi,E_0)(1+\|w(t)\|_{L^2}^2), \qquad N=2,3. \tag{3.5.8}$$

From inequalities (3.5.5)–(3.5.8) we have that for any $N \geq 2$

$$\left| \frac{d}{dt} \|w(t)\|_{L^2}^2 \right| \leq C(W, \phi, E_0)(1 + \|w(t)\|_{L^2}^2), \quad t \in (-T_{\min}, T_{\max}). \tag{3.5.9}$$

By Gronwall's lemma we conclude that

$$\|w(t)\|_{L^2} \leq C(W, \phi, E_0) e^{C(W,\phi,E_0)|t|}(1 + \|w_0\|_{L^2}), t \in (-T_{\min}, T_{\max}).$$

As we discussed before, this estimate implies (3.5.4), which finishes the proof if W is positive definite. □

Remark 3.5.1. We note that the argument given in the proof Theorem 3.1.2-(i)-(a) fails in dimension $N = 1$. In this case if we apply the Gagliardo-Nirenberg inequality to H_2, instead of (3.5.9) we obtain a bound for $\|w(t)\|_{L^2}^2$ in terms of $\|w(t)\|_{L^2}^{5/2}$, which prevents to conclude applying Gronwall's lemma.

Proof of Theorem 3.1.2-(i)-(b). In the case that W is a positive distribution, we cannot infer from (3.5.1) a uniform bound on $\|\nabla w(t)\|_{L^2}$. However, using that $W \in \mathcal{M}_{1,1}(\mathbb{R}^N)$, we will see that $\|\nabla w(t)\|_{L^2}$ can be bounded in terms of $\|w(t)\|_{L^2}$ and that we may deduce an inequality such as (3.5.9) (without assuming that $\|\nabla w(t)\|_{L^2}$ is a priori bounded). Then the conclusion follows as before.

Let $A = 4\|\phi\|_{L^\infty} + 1$. Setting

$$\begin{aligned} w_A(x,t) &= w(x,t)\chi(\{y \in \mathbb{R}^N : |w(y,t)| \leq A\})(x), \\ w_{A^c}(x,t) &= w(x,t)\chi(\{y \in \mathbb{R}^N : |w(y,t)| > A\})(x), \end{aligned}$$

where χ is the characteristic function, we deduce that $w = w_A + w_{A^c}$, $|w| = |w_A| + |w_{A^c}|$, $|w|^2 = |w_A|^2 + |w_{A^c}|^2$ and

$$\int_{\mathbb{R}^N} (W * (|\phi + w(t)|^2 - 1))(|\phi + w(t)|^2 - 1)\, dx = I_1(t) + I_2(t) + I_3(t), \tag{3.5.10}$$

with

$$\begin{aligned} I_1(t) &= \int_{\mathbb{R}^N} (W * (|\phi|^2 - 1 + 2\langle\!\langle \phi, w(t) \rangle\!\rangle))(|\phi|^2 - 1 + 2\langle\!\langle \phi, w(t) \rangle\!\rangle)\, dx \\ &\quad + 2\int_{\mathbb{R}^N} (W * |w(t)|^2)(|\phi|^2 - 1)\, dx, \\ I_2(t) &= \int_{\mathbb{R}^N} (W * |w(t)|^2)(4\langle\!\langle \phi, w_A(t) \rangle\!\rangle + |w_A(t)|^2)\, dx, \\ I_3(t) &= \int_{\mathbb{R}^N} (W * |w(t)|^2)(4\langle\!\langle \phi, w_{A^c}(t) \rangle\!\rangle + |w_{A^c}(t)|^2)\, dx. \end{aligned}$$

Notice that we have used that W is even to decompose it in terms of I_1, I_2 and I_3. Since the energy (3.5.1) is conserved in the maximal interval $(-T_{\min}, T_{\max})$, using (3.5.2) and (3.5.10), we have that for any $t \in (-T_{\min}, T_{\max})$,

$$\|\nabla w(t)\|_{L^2}^2 + I_3(t) \leq |I_1(t)| + |I_2(t)| + 4|E_0| + 2\|\nabla \phi\|_{L^2}^2. \tag{3.5.11}$$

Since W is a positive distribution, the choice of A implies that

$$\begin{aligned} I_3(t) &\geq \int_{\mathbb{R}^N} (W * |w(t)|^2)|w_{A^c}(t)|(|w_{A^c}(t)| - 4\|\phi\|_{L^\infty})\,dx \\ &\geq \int_{\mathbb{R}^N} (W * |w(t)|^2)|w_{A^c}(t)|\,dx \geq 0, \end{aligned} \tag{3.5.12}$$

so that I_3 is nonnegative. Using that $W \in \mathcal{M}_{1,1}(\mathbb{R}^N)$ we also have

$$|I_1(t)| \leq \|W\|_{2,2}(\||\phi|^2 - 1\|_{L^2} + 2\|\phi\|_{L^\infty}\|w\|_{L^2})^2 + 2\|W\|_{1,1}\|w\|_{L^2}^2(\|\phi\|_{L^\infty}^2 + 1) \tag{3.5.13}$$

and

$$|I_2(t)| \leq \|W\|_{1,1}(4A\|\phi\|_{L^\infty} + A^2)\|w(t)\|_{L^2}^2. \tag{3.5.14}$$

From inequalities (3.5.11), (3.5.13) and (3.5.14), we obtain that

$$\|\nabla w(t)\|_{L^2}^2 + I_3(t) \leq C(W, \phi, E_0)(1 + \|w(t)\|_{L^2}^2), \tag{3.5.15}$$

for any $t \in (-T_{\min}, T_{\max})$.

Let us set

$$\begin{aligned} J_1(t) &= \int_{\mathbb{R}^N} |(W * (|\phi|^2 - 1 + 2\langle\!\langle \phi, w(t)\rangle\!\rangle))w(t)|\,dx, \\ J_2(t) &= \int_{\mathbb{R}^N} |(W * |w(t)|^2)w_A(t)|\,dx, \\ J_3(t) &= \int_{\mathbb{R}^N} |(W * |w(t)|^2)w_{A^c}(t)|\,dx. \end{aligned}$$

Then the last integral in (3.5.5) is bounded by $J_1(t) + J_2(t) + J_3(t)$. As before, we conclude that

$$J_1(t) + J_2(t) \leq C(W, \phi)(1 + \|w(t)\|_{L^2}^2). \tag{3.5.16}$$

From (3.5.12) we have $J_3(t) \leq I_3(t)$. Then (3.5.15) and (3.5.12) imply that

$$J_3(t) \leq C(W, \phi, E_0)(1 + \|w(t)\|_{L^2}^2). \tag{3.5.17}$$

The estimates (3.5.16) and (3.5.17), together with (3.5.5), provide again the inequality (3.5.9), and then the proof is completed as in the previous case. □

Proof of Theorem 3.1.2-(ii). As before, the local well-posedness follows from Theorem 3.1.10. Moreover, from Theorem 3.1.2-(i)-(a) we have the global well-posedness for $N \geq 2$. From Proposition 3.2.2 we have that W is a positive definite distribution and, as shown before, this implies that $\|\nabla w(t)\|_{L^2}$ is uniformly bounded in the maximal interval $(-T_{\min}, T_{\max})$ in terms of E_0 and ϕ (see inequality (3.5.3)). Then it only remains to prove the inequality (3.1.13), for $t \in (-T_{\min}, T_{\max})$.

The argument follows the lines of the proof in [2] for the local Gross-Pitaevskii equation. For sake of completeness we give the details.

Since W is positive definite, from the conservation of energy we have

$$0 \leq \int_{\mathbb{R}^N} (W * (|\phi + w(t)|^2 - 1))(|\phi + w(t)|^2 - 1)\,dx \leq 4E_0. \tag{3.5.18}$$

On the other hand, Lemma 3.2.4 gives a lower bound for the potential energy

$$\sigma \| |\phi + w(t)|^2 - 1\|_{L^2}^2 \leq \int_{\mathbb{R}^N} (W * (|\phi + w(t)|^2 - 1))(|\phi + w(t)|^2 - 1)\, dx. \tag{3.5.19}$$

From (3.5.5) and using Hölder inequality we obtain

$$\frac{1}{2}\left|\frac{d}{dt}\|w(t)\|_{L^2}^2\right| \leq \|\Delta\phi\|_{L^2}\|w(t)\|_{L^2} + \|W\|_{2,2}\|\phi\|_{L^\infty}\||\phi + w(t)|^2 - 1\|_{L^2}\|w(t)\|_{L^2}. \tag{3.5.20}$$

Thus from (3.5.18), (3.5.19) and (3.5.20), we have that for any $\delta > 0$

$$\frac{1}{2}\left|\frac{d}{dt}(\|w(t)\|_{L^2}^2 + \delta)\right| \leq (\|w(t)\|_{L^2}^2 + \delta)^{\frac{1}{2}} \left(\|\Delta\phi\|_{L^2} + \|W\|_{2,2}\|\phi\|_{L^\infty}\sqrt{\frac{4E_0}{\sigma}}\right).$$

Dividing by $\|w(t)\|_{L^2}^2 + \delta > 0$, integrating and then taking $\delta \to 0$ we conclude that

$$\|w(t)\|_{L^2} \leq \left(\|\Delta\phi\|_{L^2} + \|W\|_{2,2}\|\phi\|_{L^\infty}\sqrt{\frac{4E_0}{\sigma}}\right)|t| + \|w_0\|_{L^2}, \tag{3.5.21}$$

for any $t \in (-T_{\min}, T_{\max})$. As discussed before, this implies that $\|w(t)\|_{H^1}$ is uniformly bounded in $(-T_{\min}, T_{\max})$. Therefore by the blow-up alternative, we infer that $T_{\min} = T_{\max} = \infty$. Since $u(t) = w(t) + \phi$ and $u_0 = w_0 + \phi$, (3.5.21) implies (3.1.13), finishing the proof. $\square$

3.6 Equation (NGP) in energy space

We recall the following results about the energy space $\mathcal{E}(\mathbb{R}^N)$. We refer to [41, 40, 39] for their proofs.

Lemma 3.6.1. *Let $u \in \mathcal{E}(\mathbb{R}^N)$. Then there exists $\phi \in C_b^\infty \cap \mathcal{E}(\mathbb{R}^N)$ with $\nabla\phi \in H^\infty(\mathbb{R}^N)$, and $w \in H^1(\mathbb{R}^N)$ such that $u = \phi + w$.*

Lemma 3.6.2. *Let $1 \leq N \leq 4$. Then $\mathcal{E}(\mathbb{R}^N)$ is a complete metric space with the distance (3.1.15), $\mathcal{E}(\mathbb{R}^N) + H^1(\mathbb{R}^N) \subset \mathcal{E}(\mathbb{R}^N)$ and the maps*

$$u \in \mathcal{E}(\mathbb{R}^N) \mapsto \nabla u \in L^2(\mathbb{R}^N),\ u \in \mathcal{E}(\mathbb{R}^N) \mapsto 1 - |u|^2 \in L^2(\mathbb{R}^N),$$
$$(u, w) \in \mathcal{E}(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \mapsto u + w \in \mathcal{E}(\mathbb{R}^N)$$

are continuous.

Lemma 3.6.3. *Assume $1 \leq N \leq 4$. Let $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$, $u \in C(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$, $v \in C(\mathbb{R}, L^2(\mathbb{R}^N))$ and*

$$\Phi(t) := \int_0^t e^{i(t-s)\Delta} u(s)(W * v(s))\, ds, \quad t \in [0, T].$$

Then $\Phi \in C([0, T], L^2(\mathbb{R}^2))$ and there exists a universal constant C such that

$$\|\Phi\|_{L^\infty((0,T),L^2)} \leq C \max\{T, T^{\frac{8-N}{8}}\}\|W\|_{2,2}(\|1-|u|^2\|_{L^\infty((0,T),L^2)} + \|\nabla u\|_{L^\infty((0,T),L^2)})\|v\|_{L^\infty((0,T),L^2)}.$$

Proof. By Lemma 1 in [41] and Lemma 3.6.2, we may decompose $u(t) = u_1(t) + u_2(t)$, with $\|u_1\|_{L^\infty(\mathbb{R},L^\infty)} \leq 3$ and

$$\|u_2\|_{L^\infty((0,T),H^1)} \leq C(\|1-|u|^2\|_{L^\infty((0,T),L^2)} + \|\nabla u\|_{L^\infty((0,T),L^2)}). \tag{3.6.1}$$

Let us set

$$\Phi_j(t) := \int_0^t e^{i(t-s)\Delta} u_j(s)(W * v(s))\, ds, \quad j = 1, 2.$$

By the Strichartz estimates we have that $\Phi_1 \in C([0,T], L^2(\mathbb{R}^2))$ and

$$\|\Phi_1\|_{L^\infty((0,T),L^2)} \leq CT\|W\|_{2,2}\|v\|_{L^\infty(\mathbb{R},L^2)}. \tag{3.6.2}$$

Since $(8/N, 4)$ is an admissible Strichartz pair in dimension $1 \leq N \leq 4$, we also infer that $\Phi_2 \in C([0,T], L^2(\mathbb{R}^2))$ and

$$\begin{aligned}\|\Phi_2\|_{L^\infty((0,T),L^2)} &\leq CT^{\frac{8-N}{8}}\|u(W*v)\|_{L^\infty(\mathbb{R},L^{4/3})} \\ &\leq CT^{\frac{8-N}{8}}\|W\|_{2,2}\|u\|_{L^\infty(\mathbb{R},L^4)}\|v\|_{L^\infty(\mathbb{R},L^2)}\end{aligned} \tag{3.6.3}$$

Combining (3.6.1)-(3.6.3) and using the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$, the conclusion follows. □

Proof of Theorem 3.1.6. After Theorem 3.1.2, the proof follows the same arguments given in [39]. For sake of completeness we sketch the proof.

Given $u_0 \in \mathcal{E}(\mathbb{R}^N)$, by Lemma 3.6.1 we have that $u_0 = \phi + \tilde{w}_0$, for some $\tilde{w}_0 \in H^1(\mathbb{R}^N)$ and ϕ satisfying (3.1.10). Thus Theorem 3.1.2 gives a solution of (NGP) of the form $u = \phi + \tilde{w}$, with $\tilde{w} \in C(\mathbb{R}, H^1(\mathbb{R}^N))$. Therefore $u = u_0 + w$, with $w = \tilde{w} - \tilde{w}_0$ is the desired solution. To prove the uniqueness in the energy space, we consider $1 \leq N \leq 4$. Let $v \in C(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ be a mild solution of (NGP) with $v(0) = u_0$. It is sufficient to show that $v - u_0 \in C(\mathbb{R}, H^1(\mathbb{R}^N))$, because then we may apply the uniqueness result given by Theorem 3.1.2. We do this by proving that $u - v \in C(\mathbb{R}, H^1(\mathbb{R}^N))$. Note that by Lemma 3.6.2, $u \in u_0 + C(\mathbb{R}, H^1(\mathbb{R}^N)) \subset C(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ and $\nabla u, \nabla v \in C(\mathbb{R}, L^2(\mathbb{R}^N))$. It only remains to prove that $u - v \in C(\mathbb{R}, L^2(\mathbb{R}^N))$. Let $T > 0$ and $t \in [0,T]$, then

$$u(t) - v(t) = i\int_0^t e^{i(t-s)\Delta}(G(u(s)) - G(v(s)))\, ds,$$

with

$$G(u) - G(v) = u(W * (|v|^2 - |u|^2)) + (u - v)(W * (1 - |v|^2)).$$

Applying Lemma 3.6.3 to $u(W * (|v|^2 - |u|^2))$ and $(u - v)(W * (1 - |v|^2))$, we conclude that $u - v \in C([0,T], L^2(\mathbb{R}^N))$. □

3.7 Other conservation laws

In this section we consider a global solution u of (NGP) given by Theorem 3.1.2. We have already seen that the energy is conserved by the flow of this solution. Now we discuss the notions of momentum and mass associated to the equation (NGP), that are also formally conserved.

3.7.1 The momentum

The vectorial momentum for (NGP) is given by

$$p(u) = \frac{1}{2}\int_{\mathbb{R}^N} \langle\langle i\nabla u, u\rangle\rangle \, dx. \tag{3.7.1}$$

A formal computation shows that the derivative of the momentum is zero and thus it is a conserved quantity. Moreover, if $u = \phi + w$ we have

$$\begin{aligned} p(u) =& \frac{1}{2}\int_{\mathbb{R}^N} \langle\langle i\nabla \phi, \phi\rangle\rangle \, dx + \frac{1}{2}\int_{\mathbb{R}^N} \langle\langle i\nabla w, w\rangle\rangle \, dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} \langle\langle i\nabla \phi, w\rangle\rangle \, dx + \frac{1}{2}\int_{\mathbb{R}^N} \langle\langle i\nabla w, \phi\rangle\rangle \, dx. \end{aligned}$$

Here the problem is that $\langle\langle i\nabla\phi, \phi - 1\rangle\rangle$ and $\langle\langle i\nabla w, \phi - 1\rangle\rangle$ are not necessarily integrable for $w \in C(\mathbb{R}, H^1(\mathbb{R}^N))$. However, a formal integration by parts yields

$$p(u) = \frac{1}{2}\int_{\mathbb{R}^N} \langle\langle i\nabla \phi, \phi\rangle\rangle \, dx + \frac{1}{2}\int_{\mathbb{R}^N} \langle\langle i\nabla w, w\rangle\rangle \, dx + \int_{\mathbb{R}^N} \langle\langle i\nabla \phi, w\rangle\rangle \, dx, \tag{3.7.2}$$

reducing the ill-defined term to $\langle\langle i\nabla\phi, \phi\rangle\rangle$, supposing that we can justify the integration by parts. In order to give a rigorous sense to these computations, we use the following definition proposed by Mariş in [77].

Definition 3.7.1. *Let $\mathcal{X}_j(\mathbb{R}^N) = \{\partial_j v : v \in \dot{H}^1(\mathbb{R}^N)\}$, with $j = 1, \dots, N$. For any $h_1 \in L^1(\mathbb{R}^N)$ and $h_2 \in \mathcal{X}_j(\mathbb{R}^N)$ we define the linear operator L_j on $L^1(\mathbb{R}^N) + \mathcal{X}_j(\mathbb{R}^N)$ by*

$$L_j(h_1 + h_2) = \frac{1}{2}\int_{\mathbb{R}^N} h_1 \; dx.$$

Lemma 3.7.2. *Let $N \geq 2$ and $j \in \{1, \dots, N\}$. Then*

$$\int_{\mathbb{R}^N} h = 0, \quad \text{for any } h \in L^1(\mathbb{R}^N) \cap \mathcal{X}_j(\mathbb{R}^N).$$

In particular L_j is a well-defined linear continuous operator on $L^1(\mathbb{R}^N) + \mathcal{X}_j(\mathbb{R}^N)$ in any dimension $N \geq 2$.

Proof. The proof of Lemma 3.7.2 is given by Mariş (Lemma 2.3 in [77]) in the case $N \geq 3$. The same argument works in dimension two, provided that a function in $\dot{H}^1(\mathbb{R}^2)$ defines a tempered distribution. In fact, this last point was shown by Gérard (see [40], p. 8), concluding the proof. □

Following the ideas proposed in [77] in dimension $N \geq 3$, we have the following result that is essential to define our notion of momentum.

Lemma 3.7.3. *Let $N \geq 2$, $j = 1, \dots, N$ and $w \in H^1(\mathbb{R}^N)$. Then $\langle\langle i\partial_j\phi, \phi\rangle\rangle \in L^1(\mathbb{R}^N) + \mathcal{X}_j(\mathbb{R}^N)$, $\langle\langle i\partial_j\phi, w\rangle\rangle \in L^1(\mathbb{R}^N)$, $\langle\langle i\phi, \partial_j w\rangle\rangle \in L^1(\mathbb{R}^N) + \mathcal{X}_j(\mathbb{R}^N)$ and*

$$L_j(\langle\langle i\partial_j\phi, w\rangle\rangle) = -L_j(\langle\langle i\phi, \partial_j w\rangle\rangle). \tag{3.7.3}$$

Proof. The assumption (3.1.10) implies that there is a radius $R > 1$ such that $|\phi(x)| \geq \frac{1}{2}$, for all $x \in B(0,R)^c$ and ϕ is C^1 in $B(0,R)^c$. Then, there are some scalar functions $\tilde{\rho}, \tilde{\theta} \in C^1(B(0,R)^c) \cap H^1_{\text{loc}}(B(0,R)^c)$ such that

$$\phi = \tilde{\rho} e^{i\tilde{\theta}}, \qquad \text{on } B(0,R)^c.$$

Moreover, since $\partial_j \phi \in L^2(\mathbb{R}^N)$ and

$$|\partial_j \phi|^2 = |\partial_j \tilde{\rho}|^2 + \tilde{\rho}^2 |\partial_j \tilde{\theta}|^2, \quad \text{on } B(0,R)^c$$

we deduce that $\partial_j \tilde{\rho}, \partial_j \tilde{\theta} \in L^2(B(0,R)^c)$. By Whitney extension theorem (cf. [68], p. 167), there exist scalar functions ρ, $\theta \in C^1(\mathbb{R}^N)$ such that $\rho = \tilde{\rho}$ and $\theta = \tilde{\theta}$ on $B(0,R)^c$. Setting

$$\phi_1 = \rho e^{i\theta} \quad \text{ and } \quad \phi_2 = \phi - \phi_1,$$

we have

$$\langle\!\langle i\partial_j \phi, \phi \rangle\!\rangle = \langle\!\langle i\partial_j \phi_1, \phi_1 \rangle\!\rangle + \langle\!\langle i\partial_j \phi_1, \phi_2 \rangle\!\rangle + \langle\!\langle i\partial_j \phi_2, \phi_1 \rangle\!\rangle + \langle\!\langle i\partial_j \phi_2, \phi_2 \rangle\!\rangle. \tag{3.7.4}$$

Since $\operatorname{supp} \phi_2, \operatorname{supp} \nabla \phi_2 \subset \bar{B}(0,R)$, the last three terms in the r.h.s. of (3.7.4) belong to $L^1(\mathbb{R}^N)$. For the remaining term, a direct computation gives

$$\langle\!\langle i\partial_j \phi_1, \phi_1 \rangle\!\rangle = -\rho^2 \partial_j \theta = (1-\rho^2)\partial_j \theta - \partial_j \theta, \quad \text{on } \mathbb{R}^N. \tag{3.7.5}$$

The fact that $\partial_j \tilde{\theta} \in L^2(B(0,R)^c)$ implies that $\partial_j \theta \in L^2(\mathbb{R}^N)$ and from (3.1.10) it follows that $|\rho|^2 - 1 \in L^2(\mathbb{R}^N)$. Therefore from (3.7.5) we conclude that $\langle\!\langle i\partial_j \phi_1, \phi_1 \rangle\!\rangle \in L^1(\mathbb{R}^N) + \mathcal{X}_j(\mathbb{R}^N)$ and hence $\langle\!\langle i\partial_j \phi, \phi \rangle\!\rangle \in L^1(\mathbb{R}^N) + \mathcal{X}_j(\mathbb{R}^N)$.

To finish the proof, we notice that from (3.1.10) and the above computations we also have that $\phi_1 \in \dot{H}^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ and $\phi_2 \in H^1(\mathbb{R}^N)$. Then a slight modification of the argument given in Lemma 2.5 in [77], allows us to deduce that $\langle\!\langle i\partial_j \phi, w \rangle\!\rangle \in L^1(\mathbb{R}^N)$, $\langle\!\langle i\phi, \partial_j w \rangle\!\rangle \in L^1(\mathbb{R}^N) + \mathcal{X}_j(\mathbb{R}^N)$ and the identity (3.7.3). $\square$

In virtue of Lemma 3.7.3 and making an analogy to (3.7.1), for $N \geq 2$ and $u \in \phi + H^1(\mathbb{R}^N)$, we define the *generalized momentum* $q = (q_1, \dots, q_N)$ as

$$q_j(u) = L_j(\langle\!\langle i\partial_j u, u \rangle\!\rangle), \quad j = 1 \dots, N.$$

Furthermore, by (3.7.3) we have

$$q_j(u) = L_j(\langle\!\langle i\partial_j \phi, \phi \rangle\!\rangle) + \frac{1}{2} \int_{\mathbb{R}^N} \langle\!\langle i\partial_j w, w \rangle\!\rangle \, dx + \int_{\mathbb{R}^N} \langle\!\langle i\partial_j \phi, w \rangle\!\rangle \, dx, \tag{3.7.6}$$

which can be seen as a rigorous formulation of (3.7.2).

In dimension one, the operator L_j is not well-defined. In fact, following the idea of the proof of Lemma 3.7.3, if we assume that $u = \rho e^{i\theta}$ then

$$\langle\!\langle iu', u \rangle\!\rangle = -\rho^2 \theta' = (1-\rho^2)\theta' - \theta'.$$

Supposing that $\lim\limits_{R \to \infty} (\theta(R) - \theta(-R))$ exists, we would have

$$\int_{\mathbb{R}} \theta'(x) \, dx = \lim_{R \to \infty} (\theta(R) - \theta(-R)). \tag{3.7.7}$$

Thus we necessarily need to modify the definition of the momentum in the one-dimensional case to take into account the phase change (3.7.7). This approach is taken in [9] using the following notion of untwisted momentum.

Definition 3.7.4. *For $u \in \phi + H^1(\mathbb{R})$, we define the operator $\mathcal{L}$ on $\phi + H^1(\mathbb{R})$ by*

$$\mathcal{L}(u) = \lim_{R\to\infty} \left(\frac{1}{2} \int_{-R}^{R} \langle\!\langle iu', u \rangle\!\rangle \, dx + \frac{1}{2} \left(\arg u(R) - \arg u(-R) \right) \right) \mod \pi \tag{3.7.8}$$

In [9] it is proved that the limit in (3.7.8) actually exists. Therefore, as in the higher dimensional case, we define the *generalized momentum in dimension one* as

$$q_1(u) = \mathcal{L}(u).$$

The following result shows that this definition gives us an analogous expression to (3.7.6).

Lemma 3.7.5 ([9])**.** *Let $u = \phi + w$, $w \in H^1(\mathbb{R})$. Then*

$$q_1(u) = \mathcal{L}(\phi) + \frac{1}{2} \int_{\mathbb{R}} \langle\!\langle iw', w \rangle\!\rangle \, dx + \int_{\mathbb{R}} \langle\!\langle i\phi', w \rangle\!\rangle \, dx.$$

Now that we have explained the notion of generalized momentum in any dimension, we can proceed to prove Theorem 3.1.8.

Proof of Theorem 3.1.8. In view of the continuous dependence of the flow, Lemma 3.7.5, (3.7.6) and Proposition 3.1.7, we only need to prove the conservation of momentum for $u_0 = \phi + w_0$, with $w_0 \in H^2(\mathbb{R}^N)$. Thus we assume that $u - \phi = w \in C(\mathbb{R}, H^2(\mathbb{R}^N)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N))$. Integrating by parts we have that for any $j = 1, \dots, N$ and $t \in \mathbb{R}$,

$$\begin{aligned} \partial_t q_j(u(t)) &= \partial_t \left(\frac{1}{2} \int_{\mathbb{R}^N} \langle\!\langle i\partial_j w(t), w(t) \rangle\!\rangle \, dx + \int_{\mathbb{R}^N} \langle\!\langle i\partial_j \phi, w(t) \rangle\!\rangle \, dx \right) \\ &= \int_{\mathbb{R}^N} \langle\!\langle i\partial_j (w(t) + \phi), \partial_t w(t) \rangle\!\rangle \, dx \\ &= \int_{\mathbb{R}^N} \langle\!\langle i\partial_j u(t), \partial_t u(t) \rangle\!\rangle \, dx \\ &= \int_{\mathbb{R}^N} \langle\!\langle \partial_j u(t), \Delta u(t) + u(t)(W * (1 - |u(t)|^2)) \rangle\!\rangle \, dx. \end{aligned}$$

Since $|\nabla u(t)|^2 \in W^{1,1}(\mathbb{R}^N)$, an integration by parts leads to

$$\begin{aligned} \partial_t q_j(u(t)) &= -\frac{1}{2} \int_{\mathbb{R}^N} \partial_j |\nabla u(t)|^2 \, dx + \int_{\mathbb{R}^N} (W * (1 - |u(t)|^2)) \langle\!\langle u(t), \partial_j u(t) \rangle\!\rangle \, dx \\ &= \int_{\mathbb{R}^N} (W * (1 - |u(t)|^2)) \langle\!\langle u(t), \partial_j u(t) \rangle\!\rangle \, dx. \end{aligned} \tag{3.7.9}$$

Now we notice that

$$\partial_j \big((1 - |u|^2)(W * (1 - |u|^2)) \big) = -2\langle\!\langle u, \partial_j u \rangle\!\rangle (W * (1 - |u|^2)) - 2(1 - |u|^2)(W * \langle\!\langle u, \partial_j u \rangle\!\rangle). \tag{3.7.10}$$

From (3.7.10) and Lemma 3.3.4, we have

$$\int_{\mathbb{R}^N} \langle\!\langle u, \partial_j u \rangle\!\rangle (W * (1 - |u|^2)) \, dx = \int_{\mathbb{R}^N} (1 - |u|^2)(W * \langle\!\langle u, \partial_j u \rangle\!\rangle) \, dx. \tag{3.7.11}$$

Since $\big((1 - |u(t)|^2)(W * (1 - |u(t)|^2)) \big) \in W^{1,1}(\mathbb{R}^N)$, from (3.7.9), (3.7.10) and (3.7.11) we infer that

$$\partial_t q_j(u(t)) = -\frac{1}{4} \int_{\mathbb{R}^N} \partial_j \big((W * (1 - |u(t)|^2))(1 - |u(t)|^2) \big) \, dx = 0,$$

concluding the proof. ∎

Remark 3.7.6. This argument also proves the conservation of momentum stated in Theorem 3.1.10.

3.7.2 The mass

In a recent article, Béthuel et al. [10] give a definition for the mass for the local Gross-Pitaevskii equation in the one-dimensional case. In this subsection we try to extend this notion to higher dimensions.

Let $\chi \in C_0^\infty(\mathbb{R};\mathbb{R})$ be a function such that $\chi(x) = 1$ if $|x| \leq 1$, $\chi(x) = 0$ if $|x| \geq 2$ and $\|\chi'\|_{L^\infty}, \|\chi''\|_{L^\infty} \leq 2$. For any $R > 0$, $a \in \mathbb{R}^N$, we set

$$\chi_{a,R}(x) = \chi\left(\frac{|x-a|}{R}\right), \quad x \in \mathbb{R}^N$$

and the quantities

$$m^+(u) = \inf_{a\in\mathbb{R}^N} \limsup_{R\to\infty} \int_{\mathbb{R}^N} (1-|u|^2)\chi_{a,R}\, dx, \quad m^-(u) = \sup_{a\in\mathbb{R}^N} \liminf_{R\to\infty} \int_{\mathbb{R}^N} (1-|u|^2)\chi_{a,R}\, dx.$$

In the case that $1-|u|^2 \in L^1(\mathbb{R}^N)$, $m^+(u) = m^-(u)$. More generally, if u is such that $m^+(u) = m^-(u)$, we define the *generalized mass* as

$$m(u) \equiv m^+(u) = m^-(u).$$

The following result is a more accurate version of Theorem 3.1.9 and shows that the generalized mass is conserved if $N \leq 4$. However, we need a faster decay for ϕ in dimensions three and four, which is at least satisfied by the travelling waves in the local problem (see [48]).

Theorem 3.7.7. *Let $1 \leq N \leq 4$. In addition to* (3.1.10)*, assume that $\nabla\phi \in L^{\frac{N}{N-1}}(\mathbb{R}^N)$ if $N = 3, 4$. Suppose that $u_0 \in \phi + H^1(\mathbb{R}^N)$ with $m^+(u_0)$ (respectively $m^-(u_0)$) finite. Then the associated solution of* (NGP) *given by Theorem 3.1.2 satisfies $m^+(u(t)) = m^+(u_0)$ (respectively $m^-(u(t)) = m^-(u_0)$), for any $t \in \mathbb{R}$. In particular, if u_0 has finite generalized mass, then the generalized mass is conserved by the flow, that is $m(u(t)) = m(u_0)$, for any $t \in \mathbb{R}$.*

Proof. Let $u_0 = \phi + w_0$ and $u = \phi + w$, $w_0 \in H^1(\mathbb{R}^N)$, $w \in C(\mathbb{R}, H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^N))$. We take a sequence $w_{0,n} \in H^2(\mathbb{R}^N)$ such that $w_{0,n} \to w_0$ in $H^1(\mathbb{R}^N)$. By Proposition 3.1.7 and the continuous dependence property of Theorem 3.1.2, the solutions $u_n = \phi + w_n$ of (NGP) with initial data $\phi + w_{0,n}$ satisfy

$$w_n \in C(\mathbb{R}, H^2(\mathbb{R}^N)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N)) \quad \text{and} \quad w_n \to w \text{ in } C(I, H^1(\mathbb{R}^N)), \tag{3.7.12}$$

for any bounded closed interval I.

Setting $\eta(t) = 1 - |u(t)|^2$, $\eta_n(t) = 1 - |u_n(t)|^2$ and using that the functions u_n are solution of (NGP), it follows

$$\partial_t \eta_n(t) = -2\,\mathrm{Re}(i\overline{u}_n(t)\Delta u_n(t)).$$

Then integrating by parts

$$\partial_t \left(\int_{\mathbb{R}^N} \eta_n(t)\chi_{a,R}\, dx\right) = \int_{\mathbb{R}^N} \partial_t\eta_n(t)\chi_{a,R}\, dx = I_1(t) + I_2(t) + I_3, \tag{3.7.13}$$

with

$$\begin{aligned}
I_1(t) &= -2\,\mathrm{Im}\int_{\mathbb{R}^N} (\overline{w}_n(t)\nabla w_n(t) + \overline{w}_n(t)\nabla\phi)\nabla\chi_{a,R}\,dx,\\
I_2(t) &= -2\,\mathrm{Im}\int_{\mathbb{R}^N} \overline{\phi}\nabla w_n(t)\nabla\chi_{a,R}\,dx,\\
I_3 &= -2\,\mathrm{Im}\int_{\mathbb{R}^N} \overline{\phi}\nabla\phi\nabla\chi_{a,R}\,dx.
\end{aligned}$$

Noticing that $\|\Delta\chi_{a,R}\|_{L^2} \leq CR^{\frac{N-4}{2}}$, we have that $\|\nabla\chi_{a,R}\|_{L^\infty}$ and $\|\Delta\chi_{a,R}\|_{L^2}$ are uniformly bounded in a and R. Setting

$$\Omega_{a,R} = \{x \in \mathbb{R}^N : R < |x-a| < 2R\}$$

and using the Cauchy-Schwarz inequality we have

$$|I_1(t)| \leq C(\phi)\|w_n(t)\|_{L^2(\Omega_{a,R})}(\|\nabla w_n(t)\|_{L^2(\Omega_{a,R})} + 1). \tag{3.7.14}$$

For I_2, we first integrate by parts

$$I_2(t) = 2\,\mathrm{Im}\int_{\mathbb{R}^N} w_n(t)(\nabla\overline{\phi}\nabla\chi_{a,R} + \overline{\phi}\Delta\chi_{a,R})\,dx,$$

thus

$$|I_2(t)| \leq C(\phi)\|w_n(t)\|_{L^2(\Omega_{a,R})}. \tag{3.7.15}$$

Using Hölder inequality, it follows that

$$|I_3| \leq \begin{cases} \|\phi\|_{L^\infty}\|\nabla\phi\|_{L^2(\Omega_{a,R})}\|\nabla\chi_{a,R}\|_{L^2}, & \text{if } N=1\\ \|\phi\|_{L^\infty}\|\nabla\phi\|_{L^{\frac{N}{N-1}}(\Omega_{a,R})}\|\nabla\chi_{a,R}\|_{L^N}, & \text{if } 2 \leq N \leq 4. \end{cases} \tag{3.7.16}$$

Note that the choice of χ implies that $\|\nabla\chi_{a,R}\|_{L^N}$ is uniformly bounded in a and R in any dimension, and so is $\|\nabla\chi_{a,R}\|_{L^2}$ in dimension one. Then by putting together (3.7.13)-(3.7.16), we obtain

$$\left|\partial_t\left(\int_{\mathbb{R}^N}\eta_n(t)\chi_{a,R}dx\right)\right| \leq C(\phi)(\|w_n(t)\|_{L^2(\Omega_{a,R})}(1+\|\nabla w_n(t)\|_{L^2}) + \|\nabla\phi\|_{L^{N^*}(\Omega_{a,R})}),$$

with $N^*=2$ if $N=1$ and $N^*=\frac{N}{N-1}$ if $2\leq N\leq 4$. Integrating this inequality between 0 and t and, by (3.7.12), passing to the limit we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}\eta(t)\chi_{a,R}\,dx - \int_{\mathbb{R}^N}\eta(0)\chi_{a,R}\,dx\right| \leq\\
&\qquad C(\phi)\int_0^{|t|}\|w(s)\|_{L^2(\Omega_{a,R})}(1+\|\nabla w(s)\|_{L^2})\,ds + C(\phi)|t|\|\nabla\phi\|_{L^{N^*}(\Omega_{a,R})}.
\end{aligned} \tag{3.7.17}$$

From the proof of Theorem 3.1.2, we deduce that for some constant K, depending only on w_0, E_0, ϕ and W,

$$\|w(t)\|_{L^2} \leq Ke^{K|t|}, \quad \|\nabla w(t)\|_{L^2} \leq Ke^{K|t|}. \tag{3.7.18}$$

Then, by Cauchy-Schwarz inequality,

$$\begin{aligned}\int_0^{|t|} \|w(s)\|_{L^2(\Omega_{a,R})}(1+\|\nabla w(s)\|_{L^2})\,ds \leq& Ke^{K|t|}\int_0^{|t|} \|w(s)\|_{L^2(\Omega_{a,R})}\,ds\\ \leq& Ke^{K|t|}|t|^{\frac{1}{2}}\left(\int_0^{|t|}\int_{\Omega_{a,R}} |w(s)|^2\,dx\,ds\right)^{\frac{1}{2}}.\end{aligned}$$

This inequality together with (3.7.18), the dominated convergence theorem and (3.7.17) imply that

$$\lim_{R\to\infty}\left(\int_{\mathbb{R}^N}(1-|u(t)|^2)\chi_{a,R}\,dx - \int_{\mathbb{R}^N}(1-|u_0|^2)\chi_{a,R}\,dx\right) = 0.$$

The conclusion follows from the definition of m^+, m^- and m. □

An interesting open question is to extend the statement of Theorem 3.1.9 to a more meaningful notion of mass such as

$$\mathfrak{m}^+(u) = \inf_{a\in\mathbb{R}}\limsup_{R\to\infty}\int_{B(a,R)}(1-|u|^2)\,dx, \quad \mathfrak{m}^-(u) = \sup_{a\in\mathbb{R}}\liminf_{R\to\infty}\int_{B(a,R)}(1-|u|^2)\,dx.$$

In fact, in the one-dimensional case, one can choose a test function χ such that

$$\|\chi_{a,R}\|_{L^2(\mathrm{supp}(\nabla\chi_{a,R}))}$$

is uniformly bounded in a and R. Then one can see that Theorem 3.1.9 remains true replacing m by $\mathfrak{m}$, recovering a result of Béthuel et al. (see Appendix in [10]). However, in higher dimensions we do not know if this is possible.

Nonexistence of traveling waves for a nonlocal Gross–Pitaevskii equation

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Chapter 4

Nonexistence of traveling waves for a nonlocal Gross–Pitaevskii equation

Abstract

We consider a Gross–Pitaevskii equation with a nonlocal interaction potential. We provide sufficient conditions on the potential such that there exists a range of speeds in which nontrivial traveling waves do not exist.

Keywords: Nonlocal Schrödinger equation, Gross–Pitaevskii equation, Traveling waves, Pohozaev identities, Nonzero conditions at infinity.

Mathematics Subject Classification 35Q55; 35Q40; 35Q51; 35B65; 37K40; 37K05; 81Q99.

4.1 Introduction

4.1.1 The problem

We consider finite energy traveling waves for the nonlocal Gross–Pitaevskii equation

$$i\partial_t u - \Delta u - u(W * (1 - |u|^2)) = 0, \quad u(x,t) \in \mathbb{C},\ x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{4.1.1}$$

Here $*$ denotes the convolution in $\mathbb{R}^N$ and W is a real-valued even distribution. The aim of this work is to provide sufficient conditions on the potential W such that these traveling waves are necessarily constant for a certain range of speeds. Equation (4.1.1) is Hamiltonian and its energy

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t)|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N} (W * (1 - |u(t)|^2))(1 - |u(t)|^2)\,dx$$

is formally conserved. A traveling wave of speed c that propagates along the x_1-axis is a solution of the form

$$u_c(x,t) = v(x_1 - ct, x_\perp), \quad x_\perp = (x_2, \dots, x_N).$$

Hence the profile v satisfies

$$ic\partial_1 v + \Delta v + v(W * (1 - |v|^2)) = 0 \text{ in } \mathbb{R}^N, \tag{NTWc}$$

and by using complex conjugation, we can restrict us to the case $c \geq 0$. Note that any constant (complex-valued) function v of modulus one verifies (NTWc), so that we refer to them as the trivial solutions.

Notice that, in the case that W coincides with the Dirac delta function, (NTWc) reduces to the classical Gross–Pitaevskii equation

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0 \text{ in } \mathbb{R}^N. \tag{TWc}$$

Equation (TWc) has been intensively studied in the last years. We refer to [7] for a survey. From now on we suppose that $N \geq 2$ and we recall the following results.

Theorem 4.1.1 ([22, 12, 47, 49]). *Let $v \in H^1_{\text{loc}}(\mathbb{R}^N)$ be a finite energy solution of* (TWc). *Assume that one of the following cases hold*

(*i*) $c = 0$.

(*ii*) $c > \sqrt{2}$.

(*iii*) $N = 2$ *and* $c = \sqrt{2}$.

Then v is a constant function of modulus one.

Theorem 4.1.2 ([12, 11, 25, 8, 77]). *There is some nonempty set $A \subset (0, \sqrt{2})$ such that for all $c \in A$ there exists a nonconstant finite energy solution of* (TWc). *Furthermore, assume that $N \geq 3$. Then there exists a nonconstant finite energy solution of* (TWc) *for all* $0 < c < \sqrt{2}$.

It would be reasonable to expect to generalize in some way these theorems to the nonlocal equation (NTWc). The aim of this paper is to investigate the analogue of Theorem 4.1.1 in the cases (i) and (ii). Before stating our precise results, we give some motivation about the critical speed.

4.1.2 Physical motivation

As explained in [31], (4.1.1) can be considered as a generalization of the equation

$$i\hbar\partial_t \Psi(x,t) = \frac{\hbar^2}{2m}\Delta\Psi(x,t) + \Psi(x,t)\int_{\mathbb{R}^N} |\Psi(y,t)|^2 V(x-y)\, dy, \text{ in } \mathbb{R}^N \times \mathbb{R}, \tag{4.1.2}$$

introduced by Gross [52] and Pitaevskii [86] to describe the kinetic of a weakly interacting Bose gas of bosons of mass m, where Ψ is the wavefunction governing the condensate in the Hartree approximation and V describes the energy interaction between bosons.

In the most typical approximation, V is considered as a Dirac delta function. Then this model has applications in several areas of physics, such as superfluidity, nonlinear optics and Bose–Einstein condensation [62, 61, 65, 26]. It seems then natural to analyze equation (4.1.2) for more general interactions. Indeed, in the study of superfluidity, supersolids and Bose–Einstein condensation, different types of nonlocal potentials have been proposed [14, 6, 32, 96, 87, 63, 103, 27, 23, 1].

Let us now proceed formally and consider a constant function u_0 of modulus one. Since (4.1.1) is invariant by a change of phase, we can assume $u_0 = 1$. Then the linearized equation of (4.1.1) at u_0 is given by

$$i\partial_t \tilde{u} - \Delta\tilde{u} + 2W * \mathrm{Re}(\tilde{u}) = 0. \tag{4.1.3}$$

Writing $\tilde{u} = \tilde{u}_1 + i\tilde{u}_2$ and taking real and imaginary parts in (4.1.3), we get

$$\begin{aligned} -\partial_t \tilde{u}_2 - \Delta\tilde{u}_1 + 2W * \tilde{u}_1 &= 0, \\ \partial_t \tilde{u}_1 - \Delta\tilde{u}_2 &= 0, \end{aligned}$$

from where we deduce that

$$\partial^2_{tt}\tilde{u} - 2W * (\Delta\tilde{u}) + \Delta^2\tilde{u} = 0. \tag{4.1.4}$$

By imposing $\tilde{u} = e^{i(\xi\cdot x - wt)}$, $w \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, as a solution of (4.1.4), we obtain the dispersion relation

$$(w(\xi))^2 = |\xi|^4 + 2\widehat{W}(\xi)|\xi|^2, \tag{4.1.5}$$

where $\widehat{W}$ denotes the Fourier transform of W. Supposing that $\widehat{W}$ is positive and continuous at the origin, we get in the long wave regime, i.e. $\xi \sim 0$,

$$w(\xi) \sim (2\widehat{W}(0))^{1/2}|\xi|.$$

Consequently, in this regime we can identify $(2\widehat{W}(0))^{1/2}$ as the speed of sound waves (also called sonic speed), so that we set

$$c_s(W) = (2\widehat{W}(0))^{1/2}.$$

The dispersion relation (4.1.5) was first observed by Bogoliubov [14] on the study of Bose–Einstein gas and under some physical considerations he established that the gas should move with a speed less than $c_s(W)$ to preserve its superfluid properties. From a mathematical point of view and comparing with Theorems 4.1.1 and 4.1.2, this encourages us to think that the nonexistence of a nontrivial solution of (NTWc) is related to the condition

$$c > c_s(W). \tag{4.1.6}$$

Actually, in Subsection 4.1.4 we provide results in this direction and in Subsection 4.1.5 we specify the discussion for some explicit potentials W which are physically relevant.

4.1.3 Hypotheses on W

Let us introduce the spaces $\mathcal{M}_{p,q}(\mathbb{R}^N)$ of tempered distributions W such that the linear operator $f \mapsto W * f$ is bounded from $L^p(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$. We will use the following hypotheses on W.

(H1) W is a real-valued even tempered distribution.

(H2) $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$. Moreover, if $N \geq 4$,

$$W \in \mathcal{M}_{N/(N-1),\infty}(\mathbb{R}^N) \cap \mathcal{M}_{2N/(N-2),\infty}(\mathbb{R}^N) \cap \mathcal{M}_{2N/(N-2),2N/(N-2)}(\mathbb{R}^N). \tag{4.1.7}$$

(H3) $\widehat{W}$ is differentiable a.e. on $\mathbb{R}^N$ and for all $j, k \in \{1, \dots, N\}$ the map $\xi \to \xi_j \partial_k \widehat{W}(\xi)$ is bounded and continuous a.e. on $\mathbb{R}^N$.

(H4) $\widehat{W} \geq 0$ a.e. on $\mathbb{R}^N$.

(H5) $\widehat{W}$ is of class C^2 in a neighborhood of the origin and $\widehat{W}(0) > 0$.

Recall that the condition $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$ is equivalent to $\widehat{W} \in L^\infty(\mathbb{R}^N)$ (see e.g. [46]). Therefore (H4) makes sense provided that (H2) holds. It is proved in [31] that under the assumptions (H1), (H2) and (H4) the Cauchy problem for (4.1.1) with nonzero condition at infinity is globally well-posed. Actually, condition (4.1.7) is more restrictive than the one used in [31] in dimension $N \geq 4$, but we need it to ensure the regularity of solutions. More precisely, in Section 4.2 we prove that under the hypothesis (H2), the solutions of (NTWc) are smooth and satisfy

$$|v(x)| \to 1, \ \nabla v(x) \to 0, \qquad \text{as } |x| \to \infty.$$

On the other hand, by Lemma 4.2.3, (4.1.7) is at least fulfilled for $W \in L^1(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$.

Assumption (H2) also implies that $E(v)$ is finite in the energy space

$$\mathcal{E}(\mathbb{R}^N) = \{\varphi \in H^1_{\text{loc}}(\mathbb{R}^N) : 1 - |\varphi|^2 \in L^2(\mathbb{R}^N), \nabla\varphi \in L^2(\mathbb{R}^N)\}.$$

Furthermore, if (H4) also holds, then by the Plancherel identity

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4(2\pi)^N}\int_{\mathbb{R}^N} \widehat{W}\, |\widehat{1 - |v|^2}|^2 \geq 0.$$

In Subsection 4.1.5 we show several examples of distributions W satisfying the conditions (H1)–(H5).

4.1.4 Statement of the results

Theorem 4.1.3. *Assume that W satisfies* (H1)–(H5). *Let $c > c_s(W)$ and suppose that there exist constants $\sigma_1, \dots, \sigma_N \in \mathbb{R}$ such that*

$$\widehat{W}(\xi) + \alpha_c \sum_{k=2}^{N} \sigma_k \xi_k \partial_k \widehat{W}(\xi) - \sigma_1 \xi_1 \partial_1 \widehat{W}(\xi) \geq 0, \text{ for a.a. } \xi \in \mathbb{R}^N, \tag{4.1.8}$$

and

$$\sum_{k=2}^{N} \sigma_k + \min\left\{-\sigma_1 - 1, \frac{\sigma_1 - 1}{\alpha_c + 2}, 2\alpha_c\sigma_j + \sigma_1 - 1\right\} \geq 0, \tag{4.1.9}$$

for all $j \in \{2, \dots, N\}$, where $\alpha_c := c^2/(c_s(W))^2 - 1$. Then nontrivial solutions of (NTWc) *in $\mathcal{E}(\mathbb{R}^N)$ do not exist.*

To apply Theorem 4.1.3 we need to verify the existence of the constants $\sigma_1, \dots, \sigma_N$ satisfying (4.1.8) and (4.1.9). To avoid this task, we provide two corollaries where the conditions for the nonexistence of traveling waves are expressed only in terms of W.

Corollary 4.1.4. *Assume that W satisfies* (H1)–(H5) *and also that*

$$\widehat{W}(\xi) \geq \max\left\{1, \frac{2}{N-1}\right\} \sum_{k=2}^{N} |\xi_k \partial_k \widehat{W}(\xi)| + |\xi_1 \partial_1 \widehat{W}(\xi)|, \text{ for a.a. } \xi \in \mathbb{R}^N. \tag{4.1.10}$$

Suppose that $c > c_s(W)$. Then nontrivial solutions of (NTWc) *in $\mathcal{E}(\mathbb{R}^N)$ do not exist.*

Corollary 4.1.5. *Assume that* W *satisfies* (H1)–(H5). *Suppose that*

$$c_s(W) < c \leq c_s(W)\left(1 + \inf_{\xi \in \mathbb{R}^N} \frac{(N-1)\widehat{W}(\xi)}{\sum_{k=2}^N |\xi_k \partial_k \widehat{W}(\xi)|}\right)^{1/2}. \tag{4.1.11}$$

Then nontrivial solutions of (NTWc) *in* $\mathcal{E}(\mathbb{R}^N)$ *do not exist.*

Concerning the static waves, we have the following result.

Theorem 4.1.6. *Assume that* W *satisfies* (H1)–(H4). *Suppose that* $c = 0$ *and that*

$$\xi_j \partial_j \widehat{W}(\xi) \leq 0, \text{ for a.a. } \xi \in \mathbb{R}^N, \tag{4.1.12}$$

for all $j \in \{1, \dots, N\}$. *Then nontrivial solutions of* (NTWc) *in* $\mathcal{E}(\mathbb{R}^N)$ *do not exist.*

Note that in the case $W = a\delta$, $a > 0$, $\widehat{W} = a$ and so that $\nabla \widehat{W} = 0$. Then conditions (4.1.10), (4.1.11) and (4.1.12) hold. Therefore, invoking Corollary 4.1.4 or 4.1.5 and Theorem 4.1.6 we obtain the nonexistence of nontrivial solutions for all

$$c \in \{0\} \cup (\sqrt{2a}, \infty). \tag{4.1.13}$$

In particular, considering $a = 1$, we recover Theorem 4.1.1 in the cases (i) and (ii).

So far, in view of (H5), we have assumed that $\widehat{W}$ is regular in a neighborhood of the origin, which in particular allows us to define $c_s(W)$. However there are interesting examples of kernels provided by the physical literature such that $\widehat{W}$ is not continuous at the origin and then $c_s(W)$ is not properly defined. For this reason we will work with a more general geometric condition on $\widehat{W}$. More precisely, denoting by $\{e_k\}_{k \in \{1,\dots,N\}}$ the canonical unitary vectors of $\mathbb{R}^N$, we introduce the function

$$w_j(\nu_1, \nu_2) := \widehat{W}(\nu_1 e_1 + \nu_2 e_j), \quad (\nu_1, \nu_2) \in \mathbb{R}^2, \ j \in \{2, \dots, N\}, \tag{4.1.14}$$

and the set

$$\Gamma_{j,c} := \{\nu = (\nu_1, \nu_2) \in \mathbb{R}^2 : |\nu|^4 + 2w_j(\nu)|\nu|^2 - c^2\nu_1^2 = 0\}.$$

Then Theorem 4.1.3 can be generalized if we replace (H5) by the condition

(H6) For all $j \in \{2, \dots, N\}$ and $c > 0$, there exist $\delta > 0$ and two functions $\gamma_{j,c}^+$ and $\gamma_{j,c}^-$, defined on the interval $(0, \delta)$, such that the set $\Gamma_{j,c} \cap B(0, \delta)$ has Lebesgue measure zero, $\gamma_{j,c}^\pm \in C^1((0, \delta))$, and

$$\gamma_{j,c}^+(t) > 0, \quad \gamma_{j,c}^-(t) < 0, \quad (t, \gamma_{j,c}^\pm(t)) \in \Gamma_{j,c}, \quad \text{for all } t \in (0, \delta).$$

Moreover, the following limits exist and are equal

$$\lim_{t \to 0^+} \left(\frac{\gamma_{j,c}^+(t)}{t}\right)^2 = \lim_{t \to 0^+} \left(\frac{\gamma_{j,c}^-(t)}{t}\right)^2 =: \ell_{j,c}.$$

Figure 4.1 illustrates condition (H6). The fact that (H5) and (4.1.6) actually imply (H6) is proved in Section 4.4 (see Lemma 4.4.1). We also note that from (H6) we infer that $\lim_{t \to 0^+} \gamma_{j,c}^\pm(t) = 0$. Moreover, if $\widehat{W}$ is even in each component, that is

$$\widehat{W}((-1)^{m_1} x_1, (-1)^{m_2} x_2, \dots, (-1)^{m_N} x_N) = \widehat{W}(x_1, x_2, \dots, x_N),$$

for all $(m_1, \dots, m_N) \in \{0, 1\}^N$, then $\gamma_{j,c}^- = -\gamma_{j,c}^+$, for all $j \in \{2, \dots, N\}$.

On the other hand, if the values $\ell_{j,c}$ are positive, a necessary condition for the existence of a nontrivial finite energy solution of (NTWc) is that they are equal.

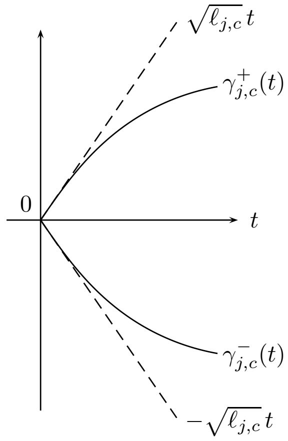

Figure 4.1: The curves $\gamma^{\pm}_{j,c}$ of condition (H6).

Lemma 4.1.7. *Let $c > 0$. Assume that W satisfies* (H1)–(H4) *and* (H6) *with $\ell_{j,c} > 0$, for all $j \in \{2, \dots, N\}$. Let $v \in \mathcal{E}(\mathbb{R}^N)$ be a nontrivial solution of* (NTWc) *in $\mathcal{E}(\mathbb{R}^N)$. Then*

$$\ell_{1,c} = \ell_{2,c} = \cdots = \ell_{N,c}.$$

Now we are ready to state our main result in its general form.

Theorem 4.1.8. *Let $c > 0$. Assume that W satisfies* (H1)–(H4) *and* (H6)*, with*

$$\ell_c := \ell_{1,c} = \ell_{2,c} = \cdots = \ell_{N,c} > 0. \tag{4.1.15}$$

Suppose that there exist constants $\sigma_1, \dots, \sigma_N \in \mathbb{R}$ such that

$$\widehat{W}(\xi) + \ell_c \sum_{k=2}^{N} \sigma_k \xi_k \partial_k \widehat{W}(\xi) - \sigma_1 \xi_1 \partial_1 \widehat{W}(\xi) \geq 0, \text{ for a.a. } \xi \in \mathbb{R}^N, \tag{4.1.16}$$

and

$$\sum_{k=2}^{N} \sigma_k + \min\left\{-\sigma_1 - 1, \frac{\sigma_1 - 1}{\ell_c + 2}, 2\ell_c\sigma_j + \sigma_1 - 1\right\} \geq 0, \tag{4.1.17}$$

for all $j \in \{2, \dots, N\}$. Then nontrivial solutions of (NTWc) *in $\mathcal{E}(\mathbb{R}^N)$ do not exist.*

Finally, we give the corresponding analogue of Corollaries 4.1.4–4.1.5.

Corollary 4.1.9. *Let $c > 0$. Assume that W satisfies* (H1)–(H4), (H6) *and* (4.1.15)*. Suppose that either* (4.1.10) *or*

$$l_c \leq \inf_{\xi \in \mathbb{R}^N} \frac{(N-1)\widehat{W}(\xi)}{\sum_{k=2}^{N} |\xi_k \partial_k \widehat{W}(\xi)|}$$

hold. Then nontrivial solutions of (NTWc) *in $\mathcal{E}(\mathbb{R}^N)$ do not exist.*

4.1.5 Examples

In this subsection we provide some potentials of physical interest for which the Cauchy problem for (4.1.1) is globally well-posed (see [31]).

(I) Given the spherically symmetric interaction of particles, in physical models it is usual to suppose that W is radial and then so is its Fourier transform, namely

$$\widehat{W}(\xi) = \rho(|\xi|),$$

for some function $\rho : [0, \infty) \to \mathbb{R}$. Assuming that ρ is differentiable, we compute

$$\xi_k \partial_k \widehat{W}(\xi) = \rho'(|\xi|)\frac{\xi_k^2}{|\xi|}, \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{4.1.18}$$

Then, using that $\sum_{k=2}^{N} \xi_k^2 = |\xi|^2 - \xi_1^2$ and that $|\xi_k| \leq |\xi|$, we obtain that conditions (4.1.10) and (4.1.11) are respectively satisfied if

$$\max\left\{1, \frac{2}{N-1}\right\} \leq \inf_{r>0} \frac{\rho(r)}{|\rho'(r)|r}, \tag{4.1.19}$$

and

$$2\rho(0) < c^2 \leq 2\rho(0)\left(1 + \inf_{r>0} \frac{\rho(r)}{|\rho'(r)|r}\right). \tag{4.1.20}$$

We consider now a generalization of the model proposed by Shchesnovich and Kraenkel [96]

$$\rho(r) = \frac{1}{(1+ar^2)^{b/2}}, \quad a, b > 0,$$

so that

$$c_s := c_s(W) = \sqrt{2}.$$

It is immediate to verify that hypotheses (H1),(H3)–(H5) are satisfied. Also, since $\widehat{W} \in L^\infty(\mathbb{R}^N)$, (H2) is fulfilled for $N = 2, 3$. Moreover, by Proposition 6.1.5 in [46], we conclude that $W \in L^1(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$ for $N \geq 4$ provided that $b > N - 1$. On the other hand,

$$\inf_{r>0} \frac{\rho(r)}{|\rho'(r)|r} = \inf_{r>0} \frac{1+ar^2}{abr^2} = \frac{1}{b}. \tag{4.1.21}$$

Therefore, using (4.1.18)–(4.1.21) and invoking Corollaries 4.1.4,4.1.5 and Theorem 4.1.6, we conclude that in the following cases there is nonexistence of nontrivial solutions of (NTWc) in $\mathcal{E}(\mathbb{R}^N)$

(a) $N = 2$, $b \leq 1/2$, $c \in (c_s, \infty)$.

(b) $N = 2$, $b > 1/2$, $c \in (c_s, \sqrt{2 + 2/b})$.

(c) $N = 3$, $b \leq 1$, $c \in (c_s, \infty)$.

(d) $N = 3$, $b > 1$, $c \in (c_s, \sqrt{2 + 2/b})$.

(e) $N \geq 4$, $b > N - 1$, $c \in (c_s, \sqrt{2 + 2/b})$.

(f) $N = 2$ or 3, $c = 0$.

(g) $N \geq 4$, $b > N - 1$, $c = 0$.

We remark that if $b \to 0$, $\widehat{W} \to 1$ and then $W \to \delta$ in a distributional sense. Thus the cases (a) and (c) could be seen as a generalization of Theorem 4.1.1 in the cases (i) and (ii).

(II) Let $N = 2, 3$ and

$$W_\varepsilon = \delta + \varepsilon f, \quad \varepsilon \geq 0,$$

where f is an even real-valued function, such that $f, |x|^2 f, |x|\nabla f \in L^1(\mathbb{R}^N)$. Then $\widehat{W_\varepsilon} = 1 + \varepsilon \widehat{f} \in C^2(\mathbb{R}^N)$. Since

$$\widehat{x_j \partial_k f} = -(\delta_{j,k} \widehat{f} + \xi_k \partial_j \widehat{f}), \tag{4.1.22}$$

we have

$$\|\widehat{f}\|_{L^\infty(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)}, \quad \|\xi_k \partial_j \widehat{f}\|_{L^\infty(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)} + \|x_j \partial_k f\|_{L^1(\mathbb{R}^N)}.$$

Then we see that W satisfies conditions (H1)–(H5) provided that $\varepsilon < \|f\|^{-1}_{L^1(\mathbb{R}^N)}$ and that the sonic speed given by

$$c_s := c_s(W) = \left(2 + 2\varepsilon \int_{\mathbb{R}^N} f\right)^{1/2},$$

is well-defined. Moreover (4.1.10) is fulfilled if

$$\varepsilon < \left(4\|f\|_{L^1(\mathbb{R}^N)} + \sum_{k=1}^N \|x_k \partial_k f\|_{L^1(\mathbb{R}^N)}\right)^{-1}. \tag{4.1.23}$$

Therefore, under condition (4.1.23), Corollary 4.1.4 implies the nonexistence of nontrivial solutions of (NTWc) in $\mathcal{E}(\mathbb{R}^N)$ for any $c \in (c_s, \infty)$.

(III) The following potential used in [23, 103] to model dipolar forces in a quantum gas yields an example in $\mathbb{R}^3$ where the speed of sound is not properly defined. Let

$$W = a\delta + bK, \qquad a, b \in \mathbb{R},$$

where K is the singular kernel

$$K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}, \quad x \in \mathbb{R}^3 \backslash \{0\}.$$

In the sequel, we will deduce from Lemma 4.1.7 and Theorem 4.1.8 that there is nonexistence of nontrivial finite energy solutions of (NTWc) in $\mathcal{E}(\mathbb{R}^N)$ for all

$$(2 \max\{a - \tilde{b}, a\})^{1/2} < c < \infty, \tag{4.1.24}$$

with $\tilde{b} = (4\pi b)/3$, provided that $a > 0$ and either

$$a \geq \tilde{b} \geq 0 \qquad \text{or} \qquad a > -2\tilde{b} \geq 0. \tag{4.1.25}$$

We now turn to the proof of condition (4.1.24). In fact, since (see [23])

$$\widehat{W}(\xi) = a + \tilde{b}\left(\frac{3\xi_3^2}{|\xi|^2} - 1\right), \qquad \xi \in \mathbb{R}^3 \backslash \{0\},$$

W satisfies (H1)–(H4) if one of the conditions in (4.1.25) holds. However, $\widehat{W}$ is not continuous at the origin. More precisely, in terms of the function defined in (4.1.14), we have that w_2 is constant equal to $a > 0$ and by Lemma 4.4.1 there exist curves $\gamma_2^{\pm}$ with $\ell_{2,c} = c^2/(2a) - 1$. On the other hand, w_3 is not continuous at the origin but assuming (4.1.24) we can explicitly solve the algebraic equation

$$(x^2 + y^2)^2 + 2w_3(x, y)(x^2 + y^2) - c^2 x^2 = 0$$

and deduce that

$$\gamma_{3,c}^{\pm}(t) = \pm\sqrt{-t^2 - a - 2\tilde{b} + \sqrt{6\tilde{b}t^2 + (a + 2\tilde{b})^2 + c^2 t^2}},$$

for $|t| < c^2 - 2(a - \tilde{b})$. Therefore (H6) holds and $\ell_{3,c} = -1 + (6\tilde{b} + c^2)/(2(a + 2\tilde{b}))$. Note that by (4.1.25), $\ell_{3,c}$ is a well-defined positive constant. By Lemma 4.1.7, a necessary condition so that the equation (NTWc) has nontrivial solutions is $\ell_{3,c} = \ell_{2,c}$, which leads us to

$$(c^2 - 3a)b = 0.$$

The case $b = 0$ has already been analyzed (see (4.1.13)). If $b \neq 0$, we obtain $c^2 = 3a$. Hence $\ell_c := \ell_{2,c} = \ell_{3,c} = 1/2$. Then, taking $\sigma_1 = 0$ and $\sigma_2 = \sigma_3 = 1/2$, (4.1.17) is satisfied and the l.h.s. of (4.1.16) reads

$$a + \tilde{b}\left(3\frac{\xi_3^2}{|\xi|^2}\left(1 - \frac{\xi_2^2}{2|\xi|^2}\right) - 1\right) + \frac{3\tilde{b}}{2}\frac{\xi_3^2}{|\xi|^2}\left(1 - \frac{\xi_3^2}{|\xi^2|}\right),$$

which is nonnegative by (4.1.25). Therefore, by Theorem 4.1.8, there is nonexistence of nontrivial solutions of (NTWc) in $\mathcal{E}(\mathbb{R}^N)$, provided that (4.1.24) and (4.1.25) hold.

As proved in [31], the Cauchy problem is also globally well-posed for other interactions such as the soft core potential

$$W(x) = \begin{cases} 1, & \text{if } |x| < a, \\ 0, & \text{otherwise,} \end{cases}$$

with $a > 0$. However, our results do not apply to this kernel, since the changes of sign of $\widehat{W}$ will prevent that an inequality such as (4.1.16) can be satisfied. Moreover, in this case the energy could be negative making the analysis more difficult. Nevertheless, $\widehat{W}$ is positive near the origin and the sonic speed is still well defined, so that it is an open question to establish which are the exact implications of change of sign of the Fourier transform in the nonexistence results.

4.1.6 Outline of the proofs and organization of the paper

We recall that Theorem 4.1.1-(i) follows from a classical Pohozaev identity. Gravejat in [47] proves Theorem 4.1.1-(ii) by combining the respective Pohozaev identity with an integral equality obtained from the Fourier analysis of the equation satisfied by $1-|v|^2$. Our results are derived in the same spirit. In the next section we prove that conditions (H1) and (H2) imply the regularity of solutions of (NTWc). In Section 4.3 we prove that condition (H6) allows us to generalize the arguments in [47] so that we can derive the integral identity (4.3.1). The fact that the set $\Gamma_{j,c}$ is described by the curves $\gamma_{j,c}^{\pm}$ is a consequence of the Morse lemma, as explained in Section 4.4.

In Section 4.5 we establish a Pohozaev identity for (NTWc) with a "remainder term" depending on the derivatives of $\widehat{W}$. Although this identity can be formally obtained for rapidly decaying functions, its proof for functions in $\mathcal{E}(\mathbb{R}^N)$ is the major technical difficulty of this paper and relies on Fourier analysis and the fact that W is even. As in [22], we then see in Section 4.6 that Theorem 4.1.6 is a straightforward consequence of this relation.

In Section 4.6 we also show that we can recast the identities described above as a suitable linear system of equations for which we can invoke the Farkas lemma to obtain the nonexistence conditions given in Theorems 4.1.8 and 4.1.3. The corollaries stated in Subsection 4.1.4 then follow by choosing the values of $\sigma_1, \dots, \sigma_N$ appropriately.

Notations. We adopt the standard notation $C(\cdot, \cdot, \dots)$ to represent a generic constant that depends only on each of its arguments. For any $x, y \in \mathbb{R}^N$, $z, w \in \mathbb{C}$, we denote the inner products in $\mathbb{R}^N$ and $\mathbb{C}$, respectively, by $x \cdot y = \sum_{i=1}^N x_i y_i$ and $\langle z, w\rangle = \operatorname{Re}(z\overline{w})$. The Kronecker delta $\delta_{k,j}$ takes the value one if $k = j$ and zero otherwise. $\mathcal{F}(f)$ or $\widehat{f}$ stand for the Fourier transform of f, namely

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-ix\cdot\xi}\, dx,$$

and $\mathcal{F}^{-1}$ for its inverse.

4.2 Regularity of solutions

From now on we fix $c \geq 0$. We assume that there exists a solution $v = v_1 + iv_2$ (v_1, v_2 real-valued) of (NTWc) in $\mathcal{E}(\mathbb{R}^N)$. We also set the real-valued functions

$$\rho := |v| = (v_1^2 + v_2^2)^{1/2}, \quad \eta := 1 - |v|^2.$$

Lemma 4.2.1. *Assume that $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$. Then $v \in W^{2,4/3}_{\text{loc}}(\mathbb{R}^N)$. Suppose further that $2 \leq N \leq 3$. Then v is smooth and bounded. Moreover, η and ∇v belong to $W^{k,p}(\mathbb{R}^N)$, for all $k \in \mathbb{N}$, $2 \leq p \leq \infty$.*

Proof. Let $\bar{x} \in \mathbb{R}^N$ and $B_r := B(\bar{x}, r)$ the ball of center $\bar{x}$ and radius r. Then

$$\|v\|_{L^4(B_1)} = \||v|^2\|_{L^2(B_1)} \leq \||v|^2 - 1\|_{L^2(\mathbb{R}^N)} + \|1\|_{L^2(B_1)} \leq E(v) + C(N). \tag{4.2.1}$$

On the other hand, we can decompose v as $v = z_1 + z_2 + z_3$, where z_1, z_2 and z_3 are the solutions of the following equations

$$\begin{cases} -\Delta z_1 = 0, & \text{in } B_1, \\ z_1 = v, & \text{on } \partial B_1, \end{cases} \tag{4.2.2}$$

$$\begin{cases} -\Delta z_2 = ic\partial_1 v, & \text{in } B_1, \\ z_2 = 0, & \text{on } \partial B_1, \end{cases} \tag{4.2.3}$$

$$\begin{cases} -\Delta z_3 = v(W * \eta), & \text{in } B_1, \\ z_3 = 0, & \text{on } \partial B_1. \end{cases} \tag{4.2.4}$$

Since z_1 is a harmonic function,

$$\|z_1\|_{C^k(B_{1/2})} \leq C(N, k, E(v)),$$

for all $k \in \mathbb{N}$. Using the Hölder inequality, (4.2.1) and elliptic regularity estimates (see e.g. [43]), we also have

$$\|z_2\|_{W^{2,2}(B_1)} \leq C(N, E(v)), \quad \|z_3\|_{W^{2,4/3}(B_1)} \leq C(N, E(v))\|\widehat{W}\|_{L^\infty(\mathbb{R}^N)}\|\eta\|_{L^2(\mathbb{R}^N)}.$$

Therefore $\|v\|_{W^{2,4/3}(B_{1/2})} \leq C(N, E(v), \eta, W)$. Furthermore, by the Sobolev embedding theorem we deduce that $\|v\|_{L^\infty(B_{1/2})}$ is bounded for $N = 2$ and then this bound holds uniformly in $\mathbb{R}^2$. If $N = 3$, we conclude that $\|v\|_{L^{12}(B_{1/2})}$ is uniformly bounded. Then using the same decomposition (4.2.2)–(4.2.4) in the ball $B_{1/4}$, identical arguments prove that $\|v\|_{W^{2,12/7}(B_{1/4})} \leq C(N, E(v), \eta, W)$, which by the Sobolev embedding theorem in dimension three implies that $\|v\|_{L^\infty(B_{1/4})}$ is uniformly bounded. Consequently, $v \in L^\infty(\mathbb{R}^N)$ for $N = 2, 3$.

Finally, using again (4.2.2)–(4.2.4) and a standard bootstrap argument, we conclude that $v \in W^{k,\infty}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$.

Now, setting $w = \partial_j v$, $j \in \{1, \dots, N\}$, and differentiating (NTWc) with respect to x_j, we obtain for any $\lambda \in \mathbb{R}$

$$L_\lambda(w) := -\Delta w - ic\partial_1 w + \lambda w = \partial_j v(W * \eta) + v(W * \partial_j \eta) + \lambda w, \quad \text{ in } \mathbb{R}^N.$$

Since $\nabla v \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, we deduce that the r.h.s. belongs to $L^2(\mathbb{R}^N)$. Then, for $\lambda > 0$ large enough, we can apply the Lax–Milgram theorem to the operator L_λ to deduce that $w \in H^2(\mathbb{R}^N)$. Thus $\nabla v \in H^2(\mathbb{R}^N)$ and a bootstrap argument shows that $\nabla v \in H^k(\mathbb{R}^N)$, for all $k \in \mathbb{N}$ and therefore, by interpolation, $\nabla v, \eta \in W^{k,p}(\mathbb{R}^N)$, for all $p \geq 2$ and $k \in \mathbb{N}$. ☐

In Lemma 4.2.1, we needed to differentiate the equation (NTWc) to improve the regularity, which required that $W * \nabla\eta$ was well-defined. If $N \geq 4$, proceeding as in Lemma 4.2.1, we can only infer that $\nabla\eta \in L^{4/3}_{\text{loc}}(\mathbb{R}^N)$ so that it is not clear that we can give a sense to the term $W * \nabla\eta$. On the other hand, if $N \geq 3$, the fact that $\nabla v \in L^2(\mathbb{R}^N)$ implies that there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $v - z_0 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ (see e.g. [59, Theorem 4.5.9]). Moreover, since (NTWc) is invariant by a change of phase, we can assume that $v - 1 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. Therefore,

$$\nabla\eta = -2\langle v - 1, \nabla v\rangle - 2\langle 1, \nabla v\rangle \in L^{N/(N-1)}(\mathbb{R}^N) + L^2(\mathbb{R}^N). \tag{4.2.5}$$

Then it would be reasonable to suppose that $W \in \mathcal{M}_{N/N-1,q}(\mathbb{R}^N)$, for some $q \geq N/N-1$. However, this is not enough to invoke the elliptic regularity estimates and that is reason why we work with the assumption (4.1.7) in (H2) if $N \geq 4$. We remark that to establish precise conditions on W that ensure the regularity of solutions of (NTWc) in higher dimensions goes beyond the scope of this paper.

Lemma 4.2.2. *Let $N \geq 4$. Assume that W satisfies* (H2). *Then v is bounded and smooth. Moreover, η and ∇v belong to $W^{k,p}(\mathbb{R}^N)$, for all $k \in \mathbb{N}$, $2 \leq p \leq \infty$.*

Proof. From (4.1.7), by duality (see e.g. [46]) we infer that $W \in \mathcal{M}_{1,N}(\mathbb{R}^N) \cap \mathcal{M}_{1,2N/(N+2)}(\mathbb{R}^N)$. Then, from the Riesz–Thorin interpolation theorem and the fact that $(1/2, (N-2)/(2N))$ and $((N-1)/N, (N-2)/(2N))$ belong to the convex hull of

$$\left\{ \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{N-1}{N}, 0\right), \left(\frac{N-2}{2N}, 0\right), \left(1, \frac{1}{N}\right), \left(1, \frac{N+2}{2N}\right) \right\},$$

we conclude that

$$W \in \mathcal{M}_{2,2N/(N-2)}(\mathbb{R}^N) \quad \text{and} \quad W \in \mathcal{M}_{N/(N-1),2N/(N-2)}(\mathbb{R}^N). \tag{4.2.6}$$

As mentioned before, we can assume that $\tilde{v} := v - 1 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. Then using (H2), (4.2.5) and (4.2.6), we are led to

$$W * \eta, W * \nabla\eta \in L^\infty(\mathbb{R}^N) \cap L^{2N/(N-2)}(\mathbb{R}^N). \tag{4.2.7}$$

Now we recast (NTWc) as

$$L_\lambda(\tilde{v}) := -\Delta\tilde{v} - ic\partial_1\tilde{v} + \lambda\tilde{v} = \tilde{v}((W * \eta) + \lambda) + W * \eta, \quad \text{in } \mathbb{R}^N, \tag{4.2.8}$$

for some $\lambda > 0$. By (4.2.7), the r.h.s. of (4.2.8) belongs to $L^{2N/(N-2)}(\mathbb{R}^N)$. Then choosing λ large enough, we can apply elliptic regularity estimates to the operator L_λ to conclude that $\tilde{v} \in W^{2,2N/(N-2)}(\mathbb{R}^N)$. Then

$$\partial_{j,k}\eta = -2(\langle v-1, \partial_{j,k}v\rangle + \langle \partial_j v, \partial_k v\rangle + \langle 1, \partial_{j,k}v\rangle) \in L^{N/(N-1)}(\mathbb{R}^N) + L^{2N/(N-2)}(\mathbb{R}^N),$$

for any $1 \leq j,k \leq N$. Therefore, by (4.1.7) and (4.2.6), $W * \partial_{j,k}\eta \in L^\infty(\mathbb{R}^N) \cap L^{2N/(N-2)}$. Thus the r.h.s. of (4.2.8) belongs to $W^{2,2N/(N-2)}(\mathbb{R}^N)$, so that $\tilde{v} \in W^{4,2N/(N-2)}(\mathbb{R}^N)$. A bootstrap argument yields that $\tilde{v} \in W^{k,2N/(N-2)}(\mathbb{R}^N)$, for any $k \in \mathbb{N}$. By the Sobolev embedding theorem, we conclude that $v \in W^{k,\infty}(\mathbb{R}^N)$ for any $k \in \mathbb{N}$. Then the conclusion follows as in Lemma 4.2.1. □

Lemma 4.2.3. *Let $W \in L^1(\mathbb{R}^N)$ if $2 \leq N \leq 3$ and $W \in L^1(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$ if $N \geq 4$. Then W fulfills* (H2).

Proof. Since $W \in L^1(\mathbb{R}^N)$, by the Young inequality we have

$$\|W * f\|_{L^p(\mathbb{R}^N)} \leq \|W\|_{L^1(\mathbb{R}^N)}\|f\|_{L^p(\mathbb{R}^N)}, \quad \text{for any } p \in [1,\infty].$$

Then, taking $p = 2$, we conclude that (H2) holds for $2 \leq N \leq 3$. For $N \geq 4$, we have $W \in L^1(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$. In particular, $W \in L^{2N/(N+2)}(\mathbb{R}^N)$ and the Young inequality implies that

$$\|W * f\|_{L^\infty(\mathbb{R}^N)} \leq \|W\|_{L^N(\mathbb{R}^N)}\|f\|_{L^{N/(N-1)}(\mathbb{R}^N)},$$
$$\|W * f\|_{L^\infty(\mathbb{R}^N)} \leq \|W\|_{L^{2N/(N+2)}(\mathbb{R}^N)}\|f\|_{L^{2N/(N-2)}(\mathbb{R}^N)}.$$

Therefore (H2) is satisfied. □

Corollary 4.2.4. *Assume that W satisfies* (H2). *Then v is smooth and bounded. Moreover, η and ∇v belong to $W^{k,p}(\mathbb{R}^N)$, for all $k \in \mathbb{N}$, $2 \leq p \leq \infty$, and*

$$\rho(x) \to 1, \ \nabla v(x) \to 0, \qquad \text{as } |x| \to \infty. \tag{4.2.9}$$

Furthermore, there exists a smooth lifting of v. More precisely, there exist $R_0 > 0$ and a smooth real-valued function θ defined on $B(0,R_0)^c$, with $\nabla\theta \in W^{k,p}(B(0,R_0)^c)$, for all $k \in \mathbb{N}$, $2 \leq p \leq \infty$, such that

$$\rho \geq \frac{1}{2} \quad \text{and} \quad v = \rho e^{i\theta} \quad \text{on } B(0,R_0)^c. \tag{4.2.10}$$

Proof. The first part is exactly Lemmas 4.2.1 and 4.2.3. In particular, v and ∇v are uniformly continuous on $\mathbb{R}^N$. Then, since $1-|v|^2 \in L^2(\mathbb{R}^N)$ and $\nabla v \in L^2(\mathbb{R}^N)$, we obtain (4.2.9). The existence of the lifting satisfying (4.2.10) follows as in [78, Proposition 2.5]. From (4.2.10) we also deduce that

$$|\nabla v|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta|^2 \quad \text{ on } B(0,R_0)^c.$$

Since $\rho \geq 1/2$ on $B(0,R_0)^c$, we infer that $\nabla\theta \in W^{k,p}(B(0,R_0)^c)$, for all $k \in \mathbb{N}$, $2 \leq p \leq \infty$. ☐

By virtue of Corollary 4.2.4, we introduce the function $\phi \in C^\infty(\mathbb{R}^N)$, $|\phi| \leq 1$, such that $\phi = 0$ on $B(0,2R_0)$ and $\phi = 1$ on $B(0,3R_0)^c$. In this way, we can assume the function $\phi\theta$ is well-defined on $\mathbb{R}^N$. This will be useful in the next section to work with global functions in terms of θ. In fact, we end this section with the following result.

Lemma 4.2.5. *Assume that W satisfies* (H2). *Then*

$$G := v_1 \nabla v_2 - v_2 \nabla v_1 - \nabla(\phi\theta), \quad \text{on } \mathbb{R}^N, \tag{4.2.11}$$

belongs to $W^{k,p}(\mathbb{R}^N)$, for all $k \in \mathbb{N}$ and $1 \leq p \leq \infty$.

Proof. By Corollary 4.2.4, $G \in C^\infty(\mathbb{R}^N)$ and moreover

$$G = -\eta \nabla \theta \quad \text{ on } B(0,3R_0)^c.$$

Since $\nabla\theta \in W^{k,p}(B(0,R_0)^c)$ and $\eta \in W^{k,p}(B(0,R_0)^c)$, for all $k \in \mathbb{N}$, $2 \leq p \leq \infty$, the conclusion follows. ☐

4.3 An integral identity

The aim of this section is to prove the following integral identity.

Proposition 4.3.1. *Let $c > 0$. Suppose that* (H2) *and* (H6) *hold with $\ell_{j,c} > 0$, for some $j \in \{2,\dots,N\}$. Then*

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta(W * \eta)) = -c \frac{\ell_{j,c}}{1+\ell_{j,c}} \int_{\mathbb{R}^N} (v_1 \partial_1 v_2 - v_2 \partial_1 v_1 - \partial_1(\phi\theta)). \tag{4.3.1}$$

We note that since W satisfies (H2), all the results of Section 4.2 hold. On the other hand, from (NTWc) we deduce that $\eta = 1 - |v|^2$ satisfies

$$\Delta \eta = -F + 2W * \eta - 2c\partial_1(\phi\theta). \tag{4.3.2}$$

where

$$F := 2|\nabla v|^2 + 2\eta(W * \eta) + 2cG_1.$$

and $G = (G_1, \dots, G_N)$ was defined in (4.2.11). Considering real and imaginary parts in (NTWc) and multiplying them by v_2 and v_1, respectively, it follows that

$$\operatorname{div}(G) = v_1 \Delta v_2 - v_2 \Delta v_1 - \Delta(\phi\theta) = \frac{c}{2}\partial_1 \eta - \Delta(\phi\theta). \tag{4.3.3}$$

Therefore, from (4.3.2) and (4.3.3), we conclude that

$$\Delta^2\eta - 2\Delta(W*\eta) + c^2\partial_{11}^2\eta = -\Delta F + 2c\partial_1(\operatorname{div} G), \quad \text{in } \mathbb{R}^N. \tag{4.3.4}$$

Since we are assuming (H2), by Corollary 4.2.4 and Lemma 4.2.5, we have that $F, G \in W^{k,1}(\mathbb{R}^N) \cap W^{k,2}(\mathbb{R}^N)$, for all $k \in \mathbb{N}$, so that (4.3.4) stands in $L^2(\mathbb{R}^N)$. Taking the Fourier transform in equation (4.3.4) and setting

$$R(\xi) := |\xi|^4 + 2\widehat{W}(\xi)|\xi|^2 - c^2\xi_1^2 \quad \text{and} \quad H(\xi) := |\xi|^2\widehat{F}(\xi) - 2c\sum_{j=1}^{N}\xi_1\xi_j\widehat{G}_j(\xi),$$

we get

$$R(\xi)\widehat{\eta}(\xi) = H(\xi), \quad \text{in } L^2(\mathbb{R}^N). \tag{4.3.5}$$

Lemma 4.3.2. *Let $c > 0$. Suppose that* (H2) *and* (H6) *hold. Then for all $j \in \{2, \dots, N\}$,*

$$H(te_1 + \gamma_{j,c}^{\pm}(t)e_j) = 0, \quad \textit{for all } t \in (0, \delta), \tag{4.3.6}$$

where δ is given by (H6).

Proof. We fix $j \in \{2, \dots, N\}$ and we prove (4.3.6) for $\gamma_{j,c}^{+}$, since the proof for $\gamma_{j,c}^{-}$ is analogous. To simplify the notation, we put $\gamma := \gamma_{j,c}^{+}$. As stated before, $F, G \in W^{k,1}(\mathbb{R}^N) \cap W^{k,2}(\mathbb{R}^N)$, for all $k \in \mathbb{N}$. In particular $F, G \in L^1(\mathbb{R}^N)$, so that $\widehat{F}, \widehat{G} \in C(\mathbb{R}^N)$. Thus H is a continuous function on $\mathbb{R}^N$.

Let $\delta > 0$ given by (H6). Arguing by contradiction, we suppose that there exist $t_0 \in (0, \delta)$ and a constant $A > 0$ such that $|H(\tilde{\xi})| \geq A$, where $\tilde{\xi} = t_0e_1 + \gamma(t_0)e_j$. By the continuity of H, there exists $r > 0$ such that $|H(\xi)| \geq A$, for all $\xi \in V_r$, where

$$V_r = B(\tilde{\xi}, r) \cap \{\alpha e_1 + \beta e_j : \alpha, \beta \in \mathbb{R}\}.$$

Thus V_r is a two-dimensional set and since $t_0 > 0$, we can choose r small enough such that $0 \notin V_r$. Then (4.3.5) yields

$$|\widehat{\eta}(\xi)|^2 \geq \frac{A^2}{(R(\xi))^2}, \quad \text{for all } \xi \in V_r \setminus \Gamma_{j,c}. \tag{4.3.7}$$

We claim that

$$I := \int_{V_r\setminus\Gamma_{j,c}} \frac{d\xi_1 d\xi_j}{(R(\xi))^2} = +\infty. \tag{4.3.8}$$

Since by hypothesis $\Gamma_{j,c} \cap B(0, \delta)$ has measure zero, (4.3.7) and (4.3.8) contradict that $\widehat{\eta} \in L^2(\mathbb{R}^N)$.

To prove (4.3.8), since V_r is a two-dimensional set, we identify it as a subset of $\mathbb{R}^2$ and so that we write e_2 instead of e_j. Then, since $\Gamma_{j,c} \cap B(0, \delta)$ has measure zero,

$$I = \int_{V_r} \frac{d\xi_1 d\xi_2}{(R(\xi))^2}.$$

To compute the integral we "straighten out" the curve γ. Namely, we introduce the change of variables

$$\begin{aligned} \xi_1 &= \nu_1 =: \Phi_1(\nu_1, \nu_2), \\ \xi_2 &= \nu_2 + \gamma(\nu_1) =: \Phi_2(\nu_1, \nu_2). \end{aligned}$$

Since γ is a C^1-function, so is Φ. Moreover, there is some set U_r such that $V_r = \Phi(U_r)$ and $|\det(J\Phi(\nu))| = 1$ for all $\nu \in U_r$. Setting $T(\nu) := R(\Phi(\nu))$, $\nu \in U_r$, the change of variables theorem yields

$$I = \int_{U_r} \frac{d\nu_1 d\nu_2}{(T(\nu))^2}. \tag{4.3.9}$$

Furthermore, since $T \in C^1(U_r)$ and $T(\nu_1, 0) = 0$ for all $(\nu_1, 0) \in U_r$, the Taylor theorem implies that for any $(\nu_1, \nu_2) \in U_r$, there is some $\bar{\nu} \in U_r$ such that

$$T(\nu_1, \nu_2) = T(\nu_1, 0) + \frac{\partial T}{\partial \nu_2}(\bar{\nu})\nu_2 = \frac{\partial T}{\partial \nu_2}(\bar{\nu})\nu_2. \tag{4.3.10}$$

On the other hand, by (H2), $\widehat{W} \in L^\infty(\mathbb{R}^N)$ and by (H3), $\nabla W \in L^\infty(V_r)$, so that $\|\widehat{W}\|_{W^{1,\infty}(V_r)} < \infty$. Thus $\|\nabla T\|_{L^\infty(U_r)} \leq C(r,\gamma)(1 + \|\widehat{W}\|_{W^{1,\infty}(V_r)})$ and from (4.3.10) we conclude that

$$|T(\nu)| \leq C(r,\gamma)(1 + \|\widehat{W}\|_{W^{1,\infty}(V_r)})|\nu_2|, \quad \text{for all } \nu \in U_r. \tag{4.3.11}$$

From (4.3.9) and (4.3.11), taking $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2) \in U_r$ such that $\tilde{\xi} = \Phi(\tilde{\nu})$ and $\varepsilon > 0$ small enough, we conclude that

$$I \geq C(r,\gamma,\widehat{W}) \int_{U_r} \frac{d\nu_1 d\nu_2}{\nu_2^2} \geq C(r,\gamma,\widehat{W}) \int_{\tilde{\nu}_1-\varepsilon}^{\tilde{\nu}_1+\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{d\nu_2 d\nu_1}{\nu_2^2} = +\infty,$$

which concludes the proof. □

Finally, we give the proof of identity (4.3.1).

Proof of Proposition 4.3.1. By Lemma 4.3.2, setting $\xi^\pm(t) = te_1 + \gamma_{j,c}^\pm(t)e_j$, we have

$$(t^2 + (\gamma_{j,c}^\pm(t))^2)\widehat{F}(\xi^\pm(t)) - 2ct^2\widehat{G}_1(\xi^\pm(t)) - 2ct\gamma_{j,c}^\pm(t)\widehat{G}_j(\xi^\pm(t)) = 0, \quad t \in (0,\delta).$$

Dividing by t^2 and passing to the limit $t \to 0^+$,

$$(1+\ell_{j,c})\widehat{F}(0) - 2c\widehat{G}_1(0) - 2c\sqrt{\ell_{j,c}}\widehat{G}_j(0) = (1+\ell_{j,c})\widehat{F}(0) - 2c\widehat{G}_1(0) + 2c\sqrt{\ell_{j,c}}\widehat{G}_j(0) = 0.$$

Therefore, since $\ell_{j,c} > 0$, $\widehat{G}_j(0) = 0$ and $(1+\ell_{j,c})\widehat{F}(0) = 2c\widehat{G}_1(0)$, which is precisely (4.3.1). □

As a consequence of Proposition 4.3.1, we obtain Lemma 4.1.7.

Proof of Lemma 4.1.7. From (4.3.1), setting

$$J(v) = \int_{\mathbb{R}^N} (|\nabla v|^2 + \eta(W * \eta)) \text{ and } P(v) = \int_{\mathbb{R}^N} (v_1\partial_1 v_2 - v_2 \partial_1 v_1 - \partial_1(\phi\theta)),$$

we infer that

$$\ell_{j,c}(J(v) + cP(v)) = -J(v). \tag{4.3.12}$$

Since v is nonconstant and $\widehat{W} \geq 0$, we have that $J(v) > 0$. Then we deduce from (4.3.12) that $J(v) + cP(v) \neq 0$ and

$$\ell_{j,c} = -\frac{J(v)}{J(v) + cP(v)}.$$

Since the r.h.s. of the equality does not depend on j, the conclusion follows. □

4.4 The set $\Gamma_{j,c}$ under the condition (H5)

In Section 4.3 we have seen that identity (4.3.1) is a consequence of the structure of the set $\Gamma_{j,c}$. More precisely, it relies on the fact that (H6) provides the existence of $\delta > 0$ and two curves $\gamma^{\pm}_{j,c}$ such that

$$\{(t, y^{\pm}(t)) : t \in (-\delta, \delta)\} \subseteq \Gamma_{j,c}.$$

If $\widehat{W}$ is of class C^2 in a neighborhood of the origin and

$$\alpha_c := \frac{c^2}{(c_s(W))^2} - 1 > 0,$$

we can use the Morse lemma to justify the existence of the curves $\gamma^{\pm}_{j,c}$ and to conclude that set $\Gamma_{j,c}$ consists of exactly these two curves near the origin. Therefore the set $\Gamma_{j,c}$ looks like Figure 4.2 and condition (H6) is fulfilled.

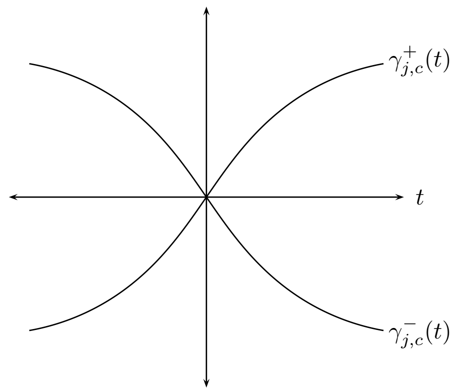

Figure 4.2: The set $\Gamma_{j,c}$ near the origin for $\widehat{W}$ of class C^2.

Lemma 4.4.1. *Assume that* (H1) *and* (H5) *hold. Assume also that* $\alpha_c > 0$. *Then, for each* $j \in \{2, \dots, N\}$, *there exist* $\delta > 0$ *and functions* $y^{\pm} \in C^1((-\delta, \delta)) \cap C^2((-\delta, \delta) \setminus \{0\})$ *such that*

$$\Gamma_{j,c} \cap B(0, \delta) = \{(t, y^{\pm}(t)) : t \in (-\delta, \delta)\}. \tag{4.4.1}$$

Moreover,

$$\lim_{t \to 0^+} y^{\pm}(t)/t = \pm\sqrt{\alpha_c}, \tag{4.4.2}$$

y^+ *is strictly increasing and* y^- *is strictly decreasing. In particular,* (H6) *is satisfied with* $l_{j,c} = \alpha_c$.

Proof. Let us set

$$R_j(\nu) := |\nu|^4 + 2w_j(\nu)|\nu|^2 - c^2\nu_1^2, \quad \nu = (\nu_1, \nu_2) \in \mathbb{R}^2.$$

In view of (H5), $R_j \in C^2(B(0, \delta_0))$, for some $\delta_0 > 0$. Since w_j is even, we have that $\partial_1 w_j(0,0) = \partial_2 w_j(0,0) = 0$. Then we obtain $R_j(0,0) = 0$, $\nabla R_j(0,0) = 0$,

$$\frac{\partial^2 R_j}{\partial \nu_1^2}(0,0) = -4\alpha_c w_j(0,0) < 0, \quad \frac{\partial^2 R_j}{\partial \nu_2^2}(0,0) = 4w_j(0,0) > 0, \quad \frac{\partial^2 R_j}{\partial \nu_1 \partial \nu_2}(0,0) = 0. \tag{4.4.3}$$

Therefore by the Morse lemma (see e.g. [82, Theorem II]) there exist two neighborhoods of the origin $U, V \subset \mathbb{R}^2$ and a local diffeomorphism $\Phi : U \to V$ such that

$$R_j(\Phi^{-1}(z)) = -2\alpha_c w_j(0,0)z_1^2 + 2w_j(0,0)z_2^2, \quad \text{for all } z = (z_1, z_2) \in V. \tag{4.4.4}$$

Moreover, denoting $\Phi = (\Phi_1, \Phi_2)$ we have for $1 \le j, k \le 2$

$$\frac{\partial \Phi_j}{\partial \nu_k}(\nu) \to \delta_{j,k}, \quad \text{as } |\nu| \to 0. \tag{4.4.5}$$

From (4.4.4) we deduce that near the origin the set of solutions of $R_j \circ \Phi^{-1} = 0$ is given by the lines

$$\{(t, \pm\sqrt{\alpha_c} t) : t \in (-\delta, \delta)\},$$

where we take $\delta > 0$ such that the set is contained in V. Since Φ is a diffeomorphism we conclude that

$$\Gamma_{j,c} \cap B(0,\delta) = \{(x_1^{\pm}(t), x_2^{\pm}(t)) : t \in (-\delta, \delta)\}, \tag{4.4.6}$$

where

$$\Phi_1(x_1^{\pm}, x_2^{\pm}) = t, \tag{4.4.7}$$

$$\Phi_2(x_1^{\pm}, x_2^{\pm}) = \pm\sqrt{\alpha_c} t. \tag{4.4.8}$$

Moreover, differentiating relation (4.4.7) with respect to t and using (4.4.5), we infer that $(x_1^{\pm})'(t) \to 1$ as $t \to 0$. Therefore we can recast (4.4.6) as in (4.4.1) with $y^{\pm} \in C^1((-\delta,\delta)) \cap C^2((-\delta,\delta) \setminus \{0\})$. Furthermore, differentiating (4.4.8) and using again (4.4.5) we conclude that

$$(y^{\pm})'(0) = \pm\sqrt{\alpha_c}.$$

Since $y^{\pm} \in C^1((-\delta,\delta))$, taking a possible smaller value δ, this implies (4.4.2) and that y^+ and y^- are strictly increasing and decreasing on $(-\delta,\delta)$, respectively. □

4.5 A Pohozaev identity

In this section we establish the following Pohozaev identity.

Proposition 4.5.1. *Assume that* (H1)–(H3) *hold. Then*

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2 + \frac{1}{4(2\pi)^N} \int_{\mathbb{R}^N} \xi_1 \partial_1 \widehat{W} |\widehat{\eta}|^2 \, d\xi, \tag{4.5.1}$$

$$E(v) = \int_{\mathbb{R}^N} |\partial_j v|^2 - \frac{c}{2} \int_{\mathbb{R}^N} (v_1\partial_1 v_2 - v_2 \partial_1 v_1 - \partial_1(\phi\theta)) + \frac{1}{4(2\pi)^N} \int_{\mathbb{R}^N} \xi_j \partial_j \widehat{W} |\widehat{\eta}|^2 \, d\xi, \tag{4.5.2}$$

for all $j \in \{2, \dots, N\}$.

Note that by Lemma 4.2.5, $G_1 = v_1\partial_1 v_2 - v_2\partial_1 v_1 - \partial_1(\phi\theta) \in L^1(\mathbb{R}^N)$, thus every integral in (4.5.1) and (4.5.2) is finite. As mentioned in Section 4.1, in the case that W is the Dirac delta function this result is well-known (see [22, 12, 47, 78]). The standard technique is to introduce a function $\chi \in C^\infty(\mathbb{R})$, with $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 2$, and $\chi_n(x) := \chi(x/n)$. Then, multiplying (NTWc) by $x_j \chi_n \partial_j \bar{v}$ and taking real part, we are led to

$$\langle ic\partial_1 v + \Delta v, x_j \chi_n \partial_j v \rangle - \frac{1}{2}(W * \eta) x_j \chi_n \partial_j \eta = 0, \quad \text{on } \mathbb{R}^N, \tag{4.5.3}$$

where we have used that

$$\langle v, \partial_j v\rangle = -\frac{1}{2}\partial_j \eta.$$

Concerning (4.5.3), we recall the following result.

Lemma 4.5.2 ([22, 12, 47, 78]). *Let $\varphi = \varphi_1 + i\varphi_2 \in \mathcal{E}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$. Assume that there exist $R^* > 0$ and a smooth real-valued function $\tilde{\theta}$ defined on $B(0, R^*)^c$, with $\nabla\tilde{\theta} \in L^2(B(0, R^*)^c)$, such that*

$$|\varphi| \geq \frac{1}{2} \quad \text{and} \quad \varphi = |\varphi|e^{i\tilde{\theta}} \quad \text{on } B(0, R_0)^c.$$

Let $\tilde{\phi} \in C^\infty(\mathbb{R}^N)$, such that $\tilde{\phi} = 0$ on $B(0, 2R^*)$ and $\tilde{\phi} = 1$ on $B(0, 3R^*)^c$. Then for all $j \in \{1, \dots, N\}$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \langle i\partial_1\varphi, x_j\chi_n\partial_j\varphi\rangle = \frac{1}{2}(1-\delta_{1,j})\int_{\mathbb{R}^N} (\varphi_1\partial_1\varphi_2 - \varphi_2\partial_1\varphi_1 - \partial_1(\tilde{\phi}\tilde{\theta})), \tag{4.5.4}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \langle \Delta\varphi, x_j\chi_n\partial_j\varphi\rangle = -\int_{\mathbb{R}^N} |\partial_j\varphi|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\varphi|^2, \tag{4.5.5}$$

$$\lim_{n\to\infty} -\frac{1}{2}\int_{\mathbb{R}^N} x_j\chi_n(1-|\varphi|^2)\partial_j(1-|\varphi|^2) = \frac{1}{4}\int_{\mathbb{R}^N}(1-|\varphi|^2)^2. \tag{4.5.6}$$

Therefore, from (4.5.3) and Lemma 4.5.2, Proposition 4.5.1 follows in the case $W = \delta$. To motivate our approach, let us briefly recall the proof of (4.5.6). First, we integrate by parts to obtain

$$\begin{aligned} A_n := & -\frac{1}{2}\int_{\mathbb{R}^N} x_j\chi_n(1-|\varphi|^2)\partial_j(1-|\varphi|^2) \\ = & \frac{1}{2}\int_{\mathbb{R}^N} \chi_n(1-|\varphi|^2)^2 + \frac{1}{2}\int_{\mathbb{R}^N} x_j\partial_j\chi_n(1-|\varphi|^2)\partial_j(1-|\varphi|^2) - A_n. \end{aligned}$$

Then, invoking the dominated convergence theorem,

$$A_n = \frac{1}{4}\int_{\mathbb{R}^N} \chi_n(1-|\varphi|^2)^2 + \frac{1}{4}\int_{\mathbb{R}^N} x_j\partial_j\chi_n(1-|\varphi|^2)\partial_j(1-|\varphi|^2) \to \frac{1}{4}\int_{\mathbb{R}^N}(1-|\varphi|^2)^2,$$

as $n \to \infty$. In particular, we see that due to a symmetry property, we can write A_n in terms of integrals to which we can apply the dominated convergence theorem. However, in our nonlocal case we cannot use this trick and we have to analyze the integral associated to the potential energy more carefully. We rely in particular on the following general result.

Proposition 4.5.3. *Let $f \in L^2(\mathbb{R}^N) \cap H^1_{\rm loc}(\mathbb{R}^N)$ be a real-valued function and $W \in \mathcal{M}_{2,2}(\mathbb{R}^N)$. Assume also that* (H1) *and* (H3) *hold. Then, for all $j \in \{1, \dots, N\}$,*

$$\lim_{n\to\infty} -\frac{1}{2}\int_{\mathbb{R}^N}(W * f)x_j\chi_n\partial_j f = \frac{1}{4}\int_{\mathbb{R}^N}(W*f)f - \frac{1}{4(2\pi)^N}\int_{\mathbb{R}^N} \xi_j\partial_j\widehat{W}(\xi)|\widehat{f}(\xi)|^2 d\xi. \tag{4.5.7}$$

The proof of Proposition 4.5.3 is rather technical, so that we postpone it. Assuming the result, we now give the proof of the Pohozaev identity.

Proof of Proposition 4.5.1 assuming Proposition 4.5.3. By putting together (4.5.3)–(4.5.5) (with $\varphi = v$) and Proposition 4.5.3, we have for $j \in \{1, \dots, N\}$,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (W * \eta)\eta = \int_{\mathbb{R}^N} |\partial_j v|^2 - (1-\delta_{1,j})\frac{c}{2}\int_{\mathbb{R}^N} (v_1\partial_1 v_2 - v_2\partial_1 v_1 - \partial_1(\chi\theta)) \\ + \frac{1}{4(2\pi)^N}\int_{\mathbb{R}^N} \xi_j \partial_j \widehat{W}(\xi)|\widehat{\eta}(\xi)|^2 d\xi,\end{aligned}$$

which is exactly (4.5.1)–(4.5.2). ◻

We remark that the main problem in order to establish the convergence in (4.5.7) is that f does not decay fast enough at infinity. Indeed, let us suppose that $x_j f, x_j\partial_j f \in L^2(\mathbb{R}^N)$. Then by the dominated convergence theorem and the Plancherel identity we have

$$B_n := -\frac{1}{2}\int_{\mathbb{R}^N} (W * f)x_j\chi_n\partial_j f \to -\frac{1}{2(2\pi)^N}\int_{\mathbb{R}^N} \widehat{W}\,\overline{\widehat{f}}\,\widehat{x_j\partial_j f}, \quad \text{as } n \to \infty.$$

Using (4.1.22), we conclude that

$$\begin{aligned}\lim_{n\to\infty} B_n &= \frac{1}{2(2\pi)^N}\int_{\mathbb{R}^N} \widehat{W}|\widehat{f}|^2 + \frac{1}{2(2\pi)^N}\int_{\mathbb{R}^N} \widehat{W}\xi_j\overline{\widehat{f}}\partial_j\widehat{f} \\ &= \frac{1}{4}\int_{\mathbb{R}^N} (W * f)f - \frac{1}{4(2\pi)^N}\int_{\mathbb{R}^N} \xi_j\partial_j\widehat{W}|\widehat{f}|^2,\end{aligned} \tag{4.5.8}$$

where we have used the Plancherel identity, integration by parts and that $\partial_j\widehat{f} \in L^2(\mathbb{R}^N)$. This yields (4.5.7), but only under these more restrictive assumptions. If we only have that $f \in L^2(\mathbb{R}^N) \cap H^1_{\text{loc}}(\mathbb{R}^N)$, we can neither invoke the dominated convergence theorem nor justify that the second integral in the r.h.s. of (4.5.8) is finite. Therefore, to deal with the limit $n \to \infty$ in Proposition 4.5.3, we first establish the following lemma.

Lemma 4.5.4. *Let $g \in L^2(\mathbb{R}^N)$ and $F \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$. Assume also that $F(\cdot, 0) \in L^\infty(\mathbb{R}^N)$ and that*

$$F(\xi, r_n) \to F(\xi, 0), \quad \text{as } |r_n| \to 0, \text{ for a.a. } \xi \in \mathbb{R}^N. \tag{4.5.9}$$

For $\varphi \in C_0^\infty(\mathbb{R}^N)$, we set

$$\widehat{\varphi}_n(\xi) := n^N\widehat{\varphi}(n\xi) \quad \text{and} \quad \Psi_n(\xi) := \int_{\mathbb{R}^N} F(\xi, r)g(\xi - r)\widehat{\varphi}_n(r)\, dr, \tag{4.5.10}$$

for a.a. $\xi \in \mathbb{R}^N$. Then

$$\Psi_n \to (2\pi)^N F(\cdot, 0)g(\cdot)\varphi(0), \quad \text{in } L^2(\mathbb{R}^N), \text{ as } n \to \infty. \tag{4.5.11}$$

Proof. Let

$$\Psi(\xi) := (2\pi)^N F(\xi, 0)g(\xi)\varphi(0), \quad \text{for a.a. } \xi \in \mathbb{R}^N.$$

We notice that by (4.5.10)

$$\int_{\mathbb{R}^N} \widehat{\varphi}_n(r)dr = \int_{\mathbb{R}^N} \widehat{\varphi}(r)dr = (2\pi)^N\varphi(0), \tag{4.5.12}$$

so that

$$\Psi_n(\xi) - \Psi(\xi) = \int_{\mathbb{R}^N} (F(\xi, r)g(\xi - r) - F(\xi, 0)g(\xi))\widehat{\varphi}_n(r)dr.$$

Then

$$\begin{aligned} |\Psi_n(\xi) - \Psi(\xi)| \leq & \|F\|_{L^\infty(\mathbb{R}^{2N})} \int_{\mathbb{R}^N} |g(\xi - r) - g(\xi)||\widehat{\varphi}_n(r)|dr \\ & + |g(\xi)| \int_{\mathbb{R}^N} |F(\xi, r) - F(\xi, 0)||\widehat{\varphi}_n(r)|dr. \end{aligned} \tag{4.5.13}$$

On the other hand, using (4.5.10) and integrating by parts, we are led to

$$\begin{aligned} |\widehat{\varphi}_n(\xi)| =& n^N \left| \int_{\mathbb{R}^N} \varphi(y) e^{-in\xi \cdot y} dy \right| \\ =& \frac{n^{N-2l}}{|\xi|^{2l}} \left| \int_{\mathbb{R}^N} \Delta^l \varphi(y) e^{-in\xi \cdot y} dy \right| \leq \frac{n^{N-2l}}{|\xi|^{2l}} \|\Delta^l \varphi\|_{L^1(\mathbb{R}^N)}, \end{aligned}$$

for any $l \in \mathbb{N}$ and any $\xi \neq 0$. Invoking this estimate for $l = N$ and the Minkowski integral inequality, we get

$$\begin{aligned} \left\| \int_{B(0,1/\sqrt{n})^c} |g(\xi - r) - g(\xi)||\widehat{\varphi}_n(r)|dr \right\|_{L^2(\mathbb{R}^N)} & \leq 2\|g\|_{L^2(\mathbb{R}^N)} \|\widehat{\varphi}_n\|_{L^1(B(0,1/\sqrt{n})^c)} \\ & \leq \frac{C(N,\varphi)}{n^{N/2}} \|g\|_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{4.5.14}$$

Similarly, we obtain

$$\left\| |g(\xi)| \int_{B(0,1/\sqrt{n})^c} |F(\xi, r) - F(\xi, 0)||\widehat{\varphi}_n(r)|dr \right\|_{L^2(\mathbb{R}^N)} \leq \frac{C(N,\varphi)}{n^{N/2}} \|F\|_{L^\infty(\mathbb{R}^{2N})} \|g\|_{L^2(\mathbb{R}^N)}. \tag{4.5.15}$$

On the other hand, using again the Minkowski integral inequality and (4.5.10),

$$\begin{aligned} & \left\| \int_{B(0,1/\sqrt{n})} |g(\xi - r) - g(\xi)||\widehat{\varphi}_n(r)|dr \right\|_{L^2(\mathbb{R}^N)} \\ & \qquad \leq \left\| \|g(\cdot - r) - g\|_{L^2(\mathbb{R}^N)} |\widehat{\varphi}_n(r)| \right\|_{L^1(B(0,1/\sqrt{n}))} \\ & \qquad \leq \sup_{|y| \leq 1/\sqrt{n}} \|g(\cdot - y) - g\|_{L^2(\mathbb{R}^N)} \|\widehat{\varphi}_n\|_{L^1(B(0,1/\sqrt{n}))} \\ & \qquad \leq \sup_{|y| \leq 1/\sqrt{n}} \|g(\cdot - y) - g\|_{L^2(\mathbb{R}^N)} \|\widehat{\varphi}\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

Since $g \in L^2(\mathbb{R}^N)$, we know that

$$\sup_{|y| \leq h} \|g(\cdot - y) - g\|_{L^2(\mathbb{R}^N)} \to 0, \text{ as } h \to 0,$$

so that

$$\left\| \int_{B(0,1/\sqrt{n})} |g(\xi - r) - g(\xi)||\widehat{\varphi}_n(r)|dr \right\|_{L^2(\mathbb{R}^N)} \to 0, \text{ as } n \to +\infty. \tag{4.5.16}$$

We now turn to the second term in the r.h.s. of (4.5.13). By a change of variables, we get that it is equal to

$$|g(\xi)| \int_{B(0,\sqrt{n})} |F(\xi, r/n) - F(\xi, 0)||\widehat{\varphi}(r)|dr. \tag{4.5.17}$$

Since $\widehat{\varphi} \in L^1(\mathbb{R}^N)$ and

$$|F(\xi, r/n) - F(\xi)||\widehat{\varphi}(r)| \leq 2\|F\|_{L^\infty(\mathbb{R}^{2N})}|\widehat{\varphi}(r)|,$$

we can deduce from (4.5.9) and the dominated convergence theorem that

$$\int_{B(0,\sqrt{n})} |F(\xi, r/n) - F(\xi)||\widehat{\varphi}(r)|dr \to 0, \text{ as } n \to +\infty,$$

for a.a. $\xi \in \mathbb{R}^N$. On the other hand,

$$|g(\xi)| \int_{B(0,\sqrt{n})} |F(\xi, r/n) - F(\xi)||\widehat{\varphi}(r)|dr \leq 2\|F\|_{L^\infty(\mathbb{R}^{2N})}\|\widehat{\varphi}\|_{L^1(\mathbb{R}^N)}|g(\xi)|,$$

Therefore, again by the dominated convergence theorem,

$$\left\| |g(\xi)| \int_{B(0,\sqrt{n})} |F(\xi, r/n) - F(\xi)||\widehat{\varphi}(r)|dr \right\|_{L^2(\mathbb{R}^N)} \to 0, \text{ as } n \to +\infty.$$

By combining with (4.5.13)–(4.5.17), we conclude (4.5.11), which finishes the proof of Lemma 4.5.4. □

Proof of Proposition 4.5.3. Setting $W_m = \mathcal{F}^{-1}(\chi_m \widehat{W}) = \mathcal{F}^{-1}(\chi_m) * W$, we have that W_m is even, $W_m \in C^\infty(\mathbb{R}^N)$,

$$\widehat{W}_m \to \widehat{W}, \ \nabla\widehat{W}_m \to \nabla\widehat{W} \text{ a.e. and } W_m * g \to W * g \text{ in } L^2(\mathbb{R}^N), \tag{4.5.18}$$

for all $g \in L^2(\mathbb{R}^N)$, as $m \to \infty$. Therefore

$$I_{n,m} := -\frac{1}{2}\int_{\mathbb{R}^N} (W_m * f)x_j\chi_n\partial_j f \underset{m\to\infty}{\longrightarrow} I_n := -\frac{1}{2}\int_{\mathbb{R}^N} \chi_n(W * f)x_j\partial_j f. \tag{4.5.19}$$

Moreover, since the Fourier transform of all derivatives of W_m have compact support, they are bounded in $L^2(\mathbb{R}^N)$. Then, by the Plancherel theorem, we conclude that

$$W_m \in W^{k,2}(\mathbb{R}^N), \quad \text{for all } k \in \mathbb{N}. \tag{4.5.20}$$

In particular, this implies that $W_m * f$ belongs to $C^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, with

$$\partial_j(W_m * f) = \partial_j W_m * f.$$

Thus, integrating by parts, we have that

$$I_{n,m} = P_{n,m} + Q_{n,m}, \tag{4.5.21}$$

where

$$P_{n,m} = \frac{1}{2}\int_{\mathbb{R}^N} (\partial_j W_m * f)x_j\chi_n f \text{ and } Q_{n,m} = \frac{1}{2}\int_{\mathbb{R}^N} (W_m * f)(\chi_n + x_j\partial_j\chi_n)f.$$

By (4.5.18),

$$\lim_{m\to\infty} Q_{n,m} = \frac{1}{2}\int_{\mathbb{R}^N} (W * f)(\chi_n + x_j\partial_j\chi_n)f. \tag{4.5.22}$$

Since $|x_j\partial_j\chi(x)| \leq 2\|\chi'\|_{L^\infty(\mathbb{R})}$, by the dominated convergence theorem,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(W*f)(\chi_n + x_j\partial_j\chi_n)f = \int_{\mathbb{R}^N}(W*f)f.$$

On the other hand, by the Cauchy–Schwarz inequality,

$$\int_{\mathbb{R}^N}|\partial_j W_m(x-y)f(y)|dy \leq \|\partial_j W_m\|_{L^2(\mathbb{R}^N)}\|f\|_{L^2(\mathbb{R}^N)}, \quad x\in\mathbb{R}^N,$$

so that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\partial_j W_m(x-y)f(y)x_jf(x)\chi_n(x)|dydx \leq 2n\|\partial_j W_m\|_{L^2(\mathbb{R}^N)}\|f\|^2_{L^2(\mathbb{R}^N)}\|\chi_n\|_{L^2(\mathbb{R}^N)}. \quad (4.5.23)$$

Since W_m is an even function, ∂W_m is odd. Then, by (4.5.23) we can use the Fubini theorem to deduce that

$$P_{n,m} = \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\partial_j W_m(x-y)f(y)f(x)\big(x_j\chi_n(x) - y_j\chi_n(y)\big)dydx, \quad (4.5.24)$$

Let us denote

$$G_{n,m}(x) := \int_{\mathbb{R}^N}\partial_j W_m(x-y)f(y)\big(x_j\chi_n(x) - y_j\chi_n(y)\big)dy, \quad (4.5.25)$$

for a.a. $x\in\mathbb{R}^N$. Arguing as before, using the Young inequality and (4.5.20), we have

$$\|G_{n,m}\|_{L^1(\mathbb{R}^N)} \leq \|\partial_j W_m\|_{L^2(\mathbb{R}^N)}\|f\|_{L^2(\mathbb{R}^N)}\|x_j\chi_n\|_{L^1(\mathbb{R}^N)} + \|\partial_j W_m\|_{L^\infty(\mathbb{R}^N)}\|fx_j\chi_n\|_{L^1(\mathbb{R}^N)},$$
$$\|G_{n,m}\|_{L^2(\mathbb{R}^N)} \leq \|\partial_j W_m\|_{L^2(\mathbb{R}^N)}\|f\|_{L^2(\mathbb{R}^N)}\|x_j\chi_n\|_{L^2(\mathbb{R}^N)} + \|\partial_j W_m\|_{L^2(\mathbb{R}^N)}\|fx_j\chi_n\|_{L^1(\mathbb{R}^N)}.$$

Thus $G_{n,m}\in L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$. Moreover, since the function $x\mapsto x_j\chi_n(x)$ is smooth on $\mathbb{R}^N$, we can write

$$x_j\chi_n(x) - y_j\chi_n(y) = \sum_{k=1}^N (x_k - y_k)\theta_k(y, x-y),$$

where

$$\theta_k(y,z) := \int_0^1\Big(\delta_{j,k}\chi_n(y+tz) + \big(y_j + tz_j\big)\partial_k\chi_n(y+tz)\Big)dt.$$

Therefore, the function $G_{n,m}$ may be written almost everywhere as

$$G_{n,m}(x) = \sum_{k=1}^N\int_{\mathbb{R}^N}(x_k - y_k)\partial_j W_m(x-y)f(y)\theta_k(y, x-y)dy,$$

so that its Fourier transform is equal to

$$\begin{aligned}\widehat{G}_{n,m}(p) &= \sum_{k=1}^N\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(x_k - y_k)\partial_j W_m(x-y)f(y)\theta_k(y,x-y)e^{-ip\cdot x}dydx\\ &= \sum_{k=1}^N\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}z_k\partial_j W_m(z)f(y)\theta_k(y,z)e^{-ip\cdot(y+z)}dydz\\ &= \frac{1}{(2\pi)^N}\sum_{k=1}^N\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}z_k\partial_j W_m(z)\widehat{f}(p-r)\tilde{\theta}_k(r,z)e^{-ip\cdot z}drdz,\end{aligned}$$

where

$$\begin{aligned}\tilde{\theta}_k(r,z) &:= \int_{\mathbb{R}^N} \theta_k(y,z)e^{-ir\cdot y}dy\\ &= \int_{\mathbb{R}^N} \Big(\int_0^1 \Big(\delta_{j,k}\chi_n(y+tz) + \big(y_j+tz_j\big)\partial_k\chi_n(y+tz)\Big)dt\Big)\theta_k(y,z)e^{-ir\cdot y}dy\\ &= \int_0^1 e^{itr\cdot z}\Big(\delta_{j,k}\widehat{\chi}_n(r) + \widehat{y_j\partial_k\chi_n}(r)\Big)dt.\end{aligned}$$

Hence, we are led to

$$\widehat{G}_{n,m}(p) = \frac{1}{(2\pi)^N}\sum_{k=1}^N \int_{\mathbb{R}^N}\int_0^1 \widehat{z_k\partial_j W_m}(p-rt)\widehat{f}(p-r)\Big(\delta_{j,k}\widehat{\chi_n}(r) + \widehat{y_j\partial_k\chi_n}(r)\Big)\,dt\,dr.$$

At this stage, we note that by (4.5.18) and (4.1.22),

$$\widehat{z_k\partial_j W_m}(p) \to \widehat{z_k\partial_j W}(p) = -p_j\partial_k\widehat{W}(p) - \delta_{k,j}\widehat{W}(p) \ \text{ a.e, as } m\to+\infty,$$

whereas

$$|\widehat{z_k\partial_j W_m}(p)| \leq \Big(1+2\|\chi'\|_{L^\infty(\mathbb{R})}\Big)\|\widehat{W}\|_{L^\infty(\mathbb{R}^N)} + \|p_j\partial_k\widehat{W}\|_{L^\infty(\mathbb{R}^N)},$$

for a.a. $p\in\mathbb{R}^N$. Invoking the dominated convergence theorem, we deduce that

$$\widehat{G}_{n,m}(p) \to \widehat{G}_n(p), \quad \text{as } m\to+\infty,$$

for a.a. $p\in\mathbb{R}^N$, where

$$\widehat{G}_n(p) := \frac{1}{(2\pi)^N}\sum_{k=1}^N \int_{\mathbb{R}^N}\int_0^1 \widehat{z_k\partial_j W}(p-rt)\widehat{f}(p-r)\Big(\delta_{j,k}\widehat{\chi}_n(r) + \widehat{y_j\partial_k\chi_n}(r)\Big)\,dt\,dr.$$

Moreover, since

$$\begin{aligned}\big|\widehat{G_{n,m}}(p)\big| \leq \frac{1}{(2\pi)^N}\sum_{k=1}^N &\Big(\big(1+2\|\chi'\|_{L^\infty(\mathbb{R})}\big)\|\widehat{W}\|_{L^\infty(\mathbb{R}^N)} + \|p_j\partial_j\widehat{W}\|_{L^\infty(\mathbb{R}^N)}\Big)\times\\ &\times \int_{\mathbb{R}^N} |\widehat{f}(p-r)|\Big|\delta_{j,k}\widehat{\chi_n}(r) + \widehat{y_j\partial_k\chi_n}(r)\Big|dr,\end{aligned}$$

it follows again from the dominated convergence theorem that

$$\widehat{G}_{n,m} \to \widehat{G}_n \text{ in } L^2(\mathbb{R}^N), \text{ as } m\to+\infty.$$

Hence, recalling (4.5.24) and (4.5.25), we are led to

$$P_{n,m} \to P_n := \frac{1}{4}\int_{\mathbb{R}^N} G_n(x)f(x)dx, \text{ as } m\to+\infty. \tag{4.5.26}$$

Finally, since

$$\begin{aligned}\widehat{\chi}_n(p) &= n^N\int_{\mathbb{R}^N}\chi_1(y)e^{-inp\cdot y}dy = n^N\widehat{\chi}_1(p),\\ \widehat{y_j\partial_k\chi_n}(p) &= n^N\int_{\mathbb{R}^N} y_j\partial_k\chi_1(y)e^{-inp\cdot y}dy = n^N\widehat{y_j\partial_k\chi_1}(np),\end{aligned}$$

$\chi_1 = 1$ and $\partial_k\chi_1 = 0$ on $B(0,1)$, applying Lemma 4.5.4 with

$$\varphi = \delta_{j,k}\chi_1 + y_j\partial_k\chi_1, \quad F(p,r) = \int_0^1 \widehat{z_k\partial_j W}(p-rt)\,dt, \quad g = \widehat{f},$$

we conclude that

$$\widehat{G}_n \to \widehat{z_j\partial_j W}\widehat{f} \quad \text{in } L^2(\mathbb{R}^N), \quad \text{as } n\to\infty. \tag{4.5.27}$$

Therefore, in view of (4.5.26), (4.5.27) and the Plancherel identity, we have

$$P_n \to \frac{1}{4(2\pi)^N}\int_{\mathbb{R}^N} \widehat{z_j\partial_j W}(p)|\widehat{f}(p)|^2 dp, \text{ as } n\to+\infty.$$

By combining with (4.1.22), (4.5.19), (4.5.21), (4.5.22) and (4.5.26), we obtain (4.5.7). $\square$

4.6 Proof of the main results

For the convenience of the reader, we first recall the Farkas Lemma (see e.g. [93]).

Lemma 4.6.1 (Farkas' Lemma). *Let $A \in \mathbb{R}^{m\times n}$ and $b\in\mathbb{R}^m$. Then one and only one of the systems*

- $Ax = b,\ \ x\geq 0,\ \ x\in\mathbb{R}^n,$
- $A^T y \geq 0,\ b^T y < 0,\ \ y \in \mathbb{R}^m,$

has a solution.

Here the notation $x\geq 0$ means that all coordinates of the vector are nonnegative. We are now in position to provide the proofs of the results stated in Subsection 4.1.4.

Proof of Theorem 4.1.8. For $j\in\{1,\dots,N\}$, let us introduce the notation

$$\mathcal{K}_j := \frac{1}{2}\int_{\mathbb{R}^N}|\partial_j v|^2, \quad \mathcal{K} := \sum_{j=1}^N \mathcal{K}_j, \quad \mathcal{R}_j := \frac{1}{4(2\pi)^N}\int_{\mathbb{R}^N}\xi_j\partial_j\widehat{W}|\widehat{\eta}|^2,$$
$$\mathcal{P} := \int_{\mathbb{R}^N}(v_1\partial_1 v_2 - v_2\partial_1 v_1 - \partial_1(\chi\theta)), \quad \mathcal{U} := \frac{1}{4}\int_{\mathbb{R}^N}(W*\eta)\eta.$$

In this way

$$E(v) = \mathcal{K} + \mathcal{U} \tag{4.6.1}$$

and Propositions 4.3.1 and 4.5.1 read

$$\mathcal{K} + 2\mathcal{U} = -\frac{c\,\ell_c}{2(1+\ell_c)}\mathcal{P}, \tag{4.6.2}$$
$$\mathcal{K} + \mathcal{U} = 2\mathcal{K}_1 + \mathcal{R}_1, \tag{4.6.3}$$
$$\mathcal{K} + \mathcal{U} = 2\mathcal{K}_j - \frac{c}{2}\mathcal{P} + \mathcal{R}_j, \tag{4.6.4}$$

for all $j \in \{2, \dots, N\}$. From (4.6.2) and (4.6.4), we obtain

$$(1 + 2\ell_c)\mathcal{K}_j + \sum_{\substack{k=1 \\ k \neq j}}^{N} \mathcal{K}_k + (\ell_c + 2)\mathcal{U} = -\ell_c \mathcal{R}_j, \quad j \in \{2, \dots, N\}. \tag{4.6.5}$$

Therefore, we can write (4.6.1), (4.6.3) and (4.6.5) as the linear system $Az = b$, with

$$z = (\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_n, \mathcal{U}), \quad b = (\mathcal{R}_1, -\ell_c \mathcal{R}_2, \dots, -\ell_c \mathcal{R}_N, E(v))$$

and $A \in \mathbb{R}^{N+1 \times N+1}$ given by

$$A_{i,j} = \begin{cases} -1, & \text{if } i = j = 1, \\ 2 + \ell_c, & \text{if } j = N+1,\ 1 < i < N+1, \\ 1 + 2\ell_c, & \text{if } i = j,\ i \neq 1, \\ 1, & \text{otherwise.} \end{cases}$$

Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N, -1)$. If $\mathcal{K} = 0$, v is constant. Therefore we suppose that $\mathcal{K} > 0$. Then using (4.1.16),

$$\begin{aligned} b^T \sigma &= \sigma_1 \mathcal{R}_1 - \ell_c \sum_{k=2}^{N} \sigma_k \mathcal{R}_k - E(v) \\ &= \frac{1}{4(2\pi)^N} \int_{\mathbb{R}^N} |\widehat{\eta}(\xi)|^2 \left(\sigma_1 \xi_1 \partial_1 \widehat{W}(\xi) - \ell_c \sum_{k=2}^{N} \sigma_k \xi_k \partial_k \widehat{W}(\xi) - \widehat{W}(\xi) \right) d\xi - \mathcal{K} \\ &\leq -\mathcal{K} < 0. \end{aligned} \tag{4.6.6}$$

On the other hand,

$$(A^T \sigma)_j = \begin{cases} -\sigma_1 + \sum_{k=2}^{N} \sigma_k - 1, & \text{if } j = 1, \\ \sigma_1 + \sum_{k=2}^{N} \sigma_k + 2\ell_c \sigma_j - 1, & \text{if } 2 \leq j \leq N, \\ \sigma_1 + (\ell_c + 2) \sum_{k=2}^{N} \sigma_k - 1, & \text{if } 2 \leq j = N+1. \end{cases}$$

Consequently, by (4.1.17), $A^T \sigma \geq 0$. However, since $z \geq 0$, this inequality together with (4.6.6) contradict Lemma 4.6.1. $\square$

Proof of Theorem 4.1.3. It is an immediate consequence of Theorem 4.1.8 and Lemma 4.4.1. $\square$

Proof of Theorem 4.1.6. Using the notation of the proof of Theorem 4.1.8, by (4.1.12) and Proposition 4.5.1 we conclude that

$$\mathcal{K} + \mathcal{U} \leq 2\mathcal{K}_j, \quad \text{for all } j \in \{1, \dots, N\}.$$

Thus, summing over j,

$$\mathcal{U} \leq \frac{2 - N}{N} \mathcal{K}. \tag{4.6.7}$$

Since $N \geq 2$, $\mathcal{K} \geq 0$ and $\mathcal{U} \geq 0$, inequality (4.6.7) implies that $\mathcal{U} = 0$ and therefore v is constant. $\square$

Proof of Corollary 4.1.4. Let us take $\sigma_1 = -1$ and $\bar{\sigma} := \sigma_2 = \cdots = \sigma_N > 0$. In order to fulfill (4.1.9), we finally fix

$$\bar{\sigma} = \max\left\{\frac{2}{(N-1)(\alpha_c+2)}, \frac{2}{N-1+\alpha_c}\right\}.$$

Then $\alpha_c\bar{\sigma} \leq \max\{1, 2/(N-1)\}$, so that

$$\widehat{W}(\xi) + \alpha_c\sum_{k=2}^{N}\sigma_k|\xi_k\partial_k\widehat{W}(\xi)| \geq \widehat{W}(\xi) - \max\left\{1, \frac{2}{N-1}\right\}\sum_{k=2}^{N}|\xi_k\partial_k\widehat{W}(\xi)| - |\xi_1\partial_1\widehat{W}(\xi)|.$$

Therefore the conclusion follows from (4.1.10) and Theorem 4.1.3. □

Proof of Corollary 4.1.5. Taking $\sigma_1 = 0$ and $\sigma_2 = \cdots = \sigma_N = 1/(N-1)$, we have that (4.1.9) is satisfied. Let

$$m := \inf_{\xi\in\mathbb{R}^N}\frac{(N-1)\widehat{W}(\xi)}{\sum_{k=2}^{N}|\xi_k\partial_k\widehat{W}(\xi)|}.$$

If $m = +\infty$, $\xi_j\partial_j\widehat{W}(\xi) = 0$ for a.a. $\xi \in \mathbb{R}^N$, for all $j \in \{2, \ldots, N\}$ and then (4.1.8) is fulfilled. If $m < \infty$, we note that (4.1.11) implies $\alpha_c \leq m$, so that

$$\widehat{W}(\xi) + \alpha_c\sum_{k=2}^{N}\sigma_k|\xi_k\partial_k\widehat{W}(\xi)| \geq \frac{m-\alpha_c}{N-1}\sum_{k=2}^{N}|\xi_k\partial_k\widehat{W}(\xi)| \geq 0.$$

Then Theorem 4.1.3 yields the conclusion. □

Proof of Corollary 4.1.9. The proof is analogous to that of Corollaries 4.1.4 and 4.1.5. The only difference is that we invoke Theorem 4.1.8 instead of Theorem 4.1.3 to conclude. □

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Traveling waves for the Landau–Lifshitz equation

Chapter 5

Traveling waves for the Landau–Lifshitz equation

Abstract

We consider finite energy traveling waves for the Landau–Lifshitz equation with easy-plane anisotropy. Using tools from the theory of harmonic maps and ideas developed for the Gross–Pitaevskii equation, we establish several properties of these solutions such as their regularity and asymptotic behavior at infinity. Our main result establishes a lower bound for the energy of nontrivial traveling waves. In particular, this provides a nonexistence result of traveling waves of small energy. In addition, in the two-dimensional case, we describe a minimizing curve which could give a variational approach to build solutions for the Landau–Lifshitz equation.

Keywords and phrases: Landau–Lifshitz equation, Harmonic maps, Schrödinger maps, Traveling waves.

5.1 Introduction

5.1.1 The problem

In this work we consider the Landau–Lifshitz equation

$$\partial_t m + m \times (\Delta m + \lambda m_3 e_3) = 0, \quad m(t,x) \in \mathbb{S}^2,\ t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{5.1.1}$$

where $e_3 = (0,0,1)$, $\lambda \in \mathbb{R}$ and $m = (m_1, m_2, m_3)$. This equation was originally introduced by L. Landau and E. Lifshitz in [72] to describe the dynamics of magnetization in a ferromagnet. Here the parameter λ takes into account the anisotropy of such material. More precisely, the value $\lambda = 0$ corresponds to the isotropic case, meanwhile $\lambda > 0$ and $\lambda < 0$ correspond to materials with an easy-axis and an easy-plane anisotropy, respectivily (see [66, 60]).

The isotropic case $\lambda = 0$ recovers the Schödinger map equation

$$\partial_t m + m \times \Delta m = 0, \tag{5.1.2}$$

which has been intensively studied due to its applications in several areas of physics and mathematics. We refer to [53] for a survey and to [3] for recent results on the global well-posedness of the Cauchy problem. In terms of localized solutions, there are several physicists' works that by means of numerical simulations and formal computations have found solutions with nontrivial topology, especially in dimension $N = 2$. Let us recall that for $N = 2$ the magnetic charge is given by

$$w(v) = \langle v, \partial_1 v \times \partial_2 v \rangle,$$

and then for any finite energy map v, constant at infinity, we can define its degree as

$$d(v) = \frac{1}{4\pi} \int_{\mathbb{R}^2} w(v)\, dx. \tag{5.1.3}$$

This quantity is an integer value that coincides with the topological degree $\mathbb{S}^2$ of the map $v \circ \Pi : \mathbb{S}^2 \to \mathbb{S}^2$, where Π refers to the stereographic projection with respect to the North Pole $(0, 0, 1)$ (see [19]). Moreover, we have

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla v(x)|^2\, dx \geq 4\pi |d(v)|, \tag{5.1.4}$$

for all $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{S}^2) \cap \dot{H}^1(\mathbb{R}^2)$, and the equality in (5.1.4) is achieved by the Belavin–Polokov instantons $Q^n = (Q^n_1, Q^n_2, Q^n_3)$,

$$Q^n_1 + iQ^n_2 = (x_1 + ix_2)^n, \; Q^n_3 = \frac{1 - (x_1^2 + x_2^2)^n}{1 + (x_1^2 + x_2^2)^n}, \quad n \in \mathbb{Z},$$

of degree $d(Q^n) = n$, that are static solutions of (5.1.2) (see [4, 89]).

However, when the material presents an anisotropy ($\lambda \neq 0$), a formal Derrick–Pohozaev scaling argument (see e.g. [33, 89, 84]) rules out the existence of static solutions in dimension $N \geq 2$. For $N = 2$, using numerical methods, B. Piette and W. Zakrzewski [85] found solitary waves periodic in time of degree n, for any $n \in \mathbb{Z}$, for the equation (5.1.1) with $\lambda > 0$. Later, this result was proved rigorously by S. Gustafson and J. Shatah [55]. Moreover, X. Pu and B. Guo [88] showed that $\lambda \neq 0$ is a necessary condition to the existence of these types of solutions.

In this paper we are interested in the case of easy-plane anisotropy $\lambda < 0$. By a scaling argument we can suppose from now on that $\lambda = -1$. Then the energy of (5.1.1) is given by

$$E(m) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla m|^2 + m_3^2 \right),$$

and it is formally conserved due to the Hamiltonian structure of (5.1.1). If m is smooth, by differentiating twice the condition $|m(t, x)|^2 = 1$ we obtain $m \cdot \Delta m = -|\nabla m|^2$, so that taking cross product of m and (5.1.1), we can recast (5.1.1) as

$$m \times \partial_t m = \Delta m + |\nabla m|^2 m - (m_3 e_3 - m_3^2 m). \tag{5.1.5}$$

Using formal developments and numerical simulations, N. Papanicolaou and P. N. Spathis [83] found in dimensions $N \in \{2, 3\}$ nonconstant finite energy traveling waves of (5.1.5), propagating with speed c along the x_1-axis, i.e. of the form

$$m_c(x, t) = u(x_1 - ct, x_2, \dots, x_N).$$

By substituting m_c in (5.1.5), the profile u satisfies

$$-\Delta u = |\nabla u|^2 u + u_3^2 u - u_3 e_3 + cu \times \partial_1 u. \tag{TW$_c$}$$

Notice that without loss of generality we can restrict us to the case $c \geq 0$. Also, we see that any constant in $\mathbb{S}^1 \times \{0\}$ satisfies (TW$_c$), so that we refer to them as the trivial solutions. Since we are interested in finite energy solutions, the natural energy space to work in is

$$\mathcal{E}(\mathbb{R}^N) = \{v \in L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^3) : \nabla v \in L^2(\mathbb{R}^N),\ v_3 \in L^2(\mathbb{R}^N),\ |v| = 1 \text{ a.e. on } \mathbb{R}^N\}.$$

Equation (TW$_c$) corresponds formally to the Euler–Lagrange equation associated to the problem of minimizing the energy for a fixed momentum (see Subsection 5.1.2).

In [83] the authors determine a branch of nontrivial solutions of (TW$_c$), axisymmetric around the axis x_1, for any speed $c \in (0, 1)$ and they also conjectured that there is no nontrivial solution for $c \geq 1$. Although the total charge of these solutions is zero ($d(u) = 0$), they have areas with nontrivial topology. More precisely, in the two-dimensional case, they compute a critical speed $c^* \approx 0.78$ such that for any $c \in (0, c^*)$, there are exactly two points $q^{\pm} = (\pm a_c, 0)$ ($a_c > 0$), such that $u_3(q^{\pm}) = 1$ and $|u_3| < 1$ on $\mathbb{R}^2 \setminus \{q^{\pm}\}$. Moreover, far away these points $|u_3|$ is almost zero and the function u covers the upper hemisphere of $\mathbb{S}^2$. Furthermore, using the stereographic variable

$$\psi = \frac{u_1 + iu_2}{1 + u_3},$$

we have that ψ satisfies

$$\Delta\psi + \frac{1 - |\psi|^2}{1 + |\psi|^2}\psi - ic\partial_1\psi = \frac{2\bar{\psi}}{1 + |\psi|^2}(\nabla\psi \cdot \nabla\psi),$$

that seems like a perturbed equation for the traveling waves for the Gross–Pitaevskii equation (see [7]). The function ψ vanishes only at $q^{\pm}$ and around each point $q^{\pm}$ the degree $\mathbb{S}^1$ of $\psi/|\psi|$ is

$$\deg\left(\frac{\psi}{|\psi|}, \partial B(q^{\pm}, r), \mathbb{S}^1\right) = \frac{1}{2\pi}\int_{\partial B(q^{\pm}, r)} \partial_\tau \phi^{\pm} = \pm 1,$$

for $r > 0$ small, where $\psi = |\psi|e^{i\phi^{\pm}}$ on $\partial B(q^{\pm}, r)$ and ∂_τ is the tangential derivative. Therefore the function ψ has two vortices at $q^{\pm}$ of degree ± 1.

For speeds $c \in (c^*, 1)$, we have that $\|u_3\|_{L^\infty(\mathbb{R}^2)} < 1$ and then the solution has no vortices. Using the dispersion curve of energy as a function of the first component of the (vectorial) momentum, i.e.

$$p(u) = -\int_{\mathbb{R}^2} x_2 w(u),$$

these solutions are depicted in Figure 5.1. In particular, we see that the curve has a nonzero minimum.

The three-dimensional case is similar with a critical value $c^* \approx 0,93$ such that the solutions of (TW$_c$) have this time a vortex ring structure for $c < c^*$. The design of the energy-momentum curve in dimension three is also similar to Figure 5.1.

This type of solutions look like to those found by C. A. Jones, S. J. Putterman and P. H. Roberts for the Gross–Pitaevskii equation [62, 61] and studied from a mathematical point of view in [8, 77]. It is important to recall that the energy-momentum curve for the Gross–Pitaevskii equation tends

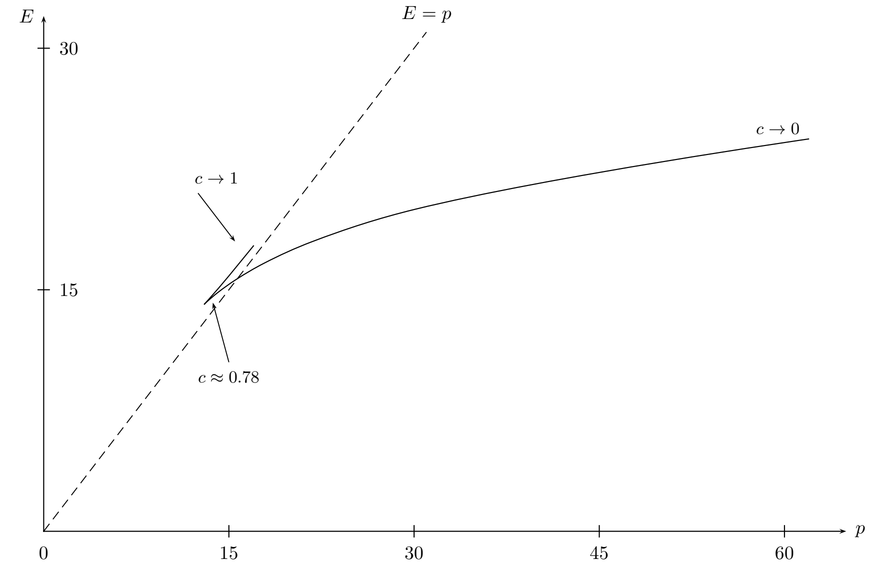

Figure 5.1: Curve of energy E as a function of the momentum p in the two-dimensional case.

to zero as $p \to 0$ in the two-dimensional case. This constitutes a fundamental difference with the curve for the solutions of the Landau–Lifshitz equation, that remains far from the origin (see Figure 5.1). One of the main objectives of this paper is to establish rigorously that the curve of minimizing solutions is far from the origin (see Theorem 5.1.10).

Recently, F. Lin and J. Wei [74] proved the existence of nontrivial finite energy solutions of the equation (TW_c) for small values of c in dimension two and three by perturbative arguments. Another approach to show their existence might be to consider the problem of minimization of energy under the constraint of momentum, as will be discussed in the next subsection.

In the case $N = 1$, (TW_c) is completely integrable and we can compute the solutions in $\mathcal{E}(\mathbb{R})$ explicitly (see Section 5.9). Moreover, expressing the energy in terms of the momentum of each nontrivial solution, we obtain Figure 5.2. In particular we note that there are solutions of small energy, but there is a maximum value for the energy and the momentum, in order to have nontrivial solutions.

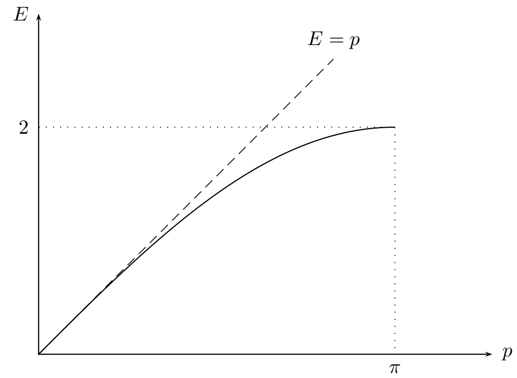

Figure 5.2: Curve of energy E as a function of the momentum p in the one-dimensional case.

5.1.2 The momentum and the minimization curve

The vectorial momentum $P = (P_1, P_2)$ (in dimension two) is given by

$$P_1(v) = -\int_{\mathbb{R}^2} x_2 w(v)\, dx, \quad P_2(v) = \int_{\mathbb{R}^2} x_1 w(v)\, dx, \tag{5.1.6}$$

which, at least formally, is conserved by the flow of the equation (5.1.5). However the momentum is not well-defined in $\mathcal{E}(\mathbb{R}^2)$, since the map $v \in \mathcal{E}(\mathbb{R}^2) \to x_2 w(v) \in \mathbb{R}$ is not necessarily integrable. Let us suppose that there is $R \equiv R(v)$ such that we have the lifting (see Lemma 5.2.4)

$$\check{v} \equiv v_1 + iv_2 = \varrho e^{i\theta}, \text{ on } B(0,R)^c, \tag{5.1.7}$$

where $\varrho \equiv \sqrt{v_1^2 + v_2^2} = \sqrt{1 - v_3^2}$ and $\varrho, \theta \in \dot{H}^1(B(0,R)^c)$. It follows that

$$w(v) = -\operatorname{curl}(v_3 \nabla\theta),$$

where $\operatorname{curl}(f_1, f_2) = \partial_1 f_2 - \partial_2 f_1$. Assuming that the lifting holds on $\mathbb{R}^2$, a formal integration by parts yields

$$P_1(v) = \int_{\mathbb{R}^2} v_3 \partial_1\theta, \quad P_2(v) = \int_{\mathbb{R}^2} v_3 \partial_2\theta. \tag{5.1.8}$$

Since for $j \in \{1, 2\}$,

$$|v_3 \partial_j \theta| \le \frac{|v_3||1 - v_3^2|^{\frac{1}{2}} |\partial_j\theta|}{(1 - \|v_3\|^2_{L^\infty(\mathbb{R}^2)})^{1/2}} \le \frac{e(v)}{(1 - \|v_3\|^2_{L^\infty(\mathbb{R}^2)})^{1/2}}, \tag{5.1.9}$$

where $e(v)$ is the energy density

$$e(v) \equiv \frac{1}{2}(|\nabla v|^2 + v_3^2) = \frac{1}{2}\Big(\frac{|\nabla v_3|^2}{1 - v_3^2} + \big(1 - v_3^2\big)|\nabla\theta|^2 + v_3^2\Big),$$

it follows that the expression for the momentum (5.1.8) is well-defined when the function has a global lifting. However the existence of the lifting is a matter of topological nature. In fact, this issue is related to the degree $\mathbb{S}^1$ of the application

$$\frac{\check{v}}{|\check{v}|} : \partial B(0,R) \to \mathbb{S}^1,$$

provide that $\check{v}/|\check{v}|$ is well-defined. Note that these types of problems have also been encountered in the study of the traveling waves for the Gross–Pitaevskii equation (see [7, 77, 10, 31]). To overcome this drawback, we can consider the space

$$\tilde{\mathcal{E}}(\mathbb{R}^2) = \{v \in \mathcal{E}(\mathbb{R}^2) : \exists R \ge 0 \text{ s.t. } \|v_3\|_{L^\infty(B(0,R))^c)} < 1\},$$

so that for all $v \in \tilde{\mathcal{E}}(\mathbb{R}^2)$ there exists $R \equiv R(v)$ such that the lifting (5.1.7) holds. In addition, we provide a notion of *generalized momentum* valid for all $v \in \tilde{\mathcal{E}}(\mathbb{R}^2)$. On the other hand, we will see that any solution of (TW_c) in $\mathcal{E}(\mathbb{R}^2)$ also belongs to $\tilde{\mathcal{E}}(\mathbb{R}^2)$ (see Corollary 5.2.5) and then this definition will be sufficiently general for our scopes.

Another difficulty is that (5.1.6) is not invariant under translations. In fact, setting $p \equiv P_1$ and using the translation function τ_a defined by

$$\tau_a f(\cdot) = f(\cdot - a), \quad a = (a_1, a_2) \in \mathbb{R}^2,$$

we have

$$p(\tau_a u) = p(u) - 4\pi a_2 d(u). \tag{5.1.10}$$

However, the traveling waves found in [83] have degree zero ($d(u) = 0$), so we could restrict us to this type of solutions.

At least formally, a possible method to construct a solution $u \in \tilde{\mathcal{E}}(\mathbb{R}^2)$ for (TW$_c$), with a prescribed momentum $p(u) = \mathfrak{p}$ is to consider the minimization problem

$$\inf \big\{ E(v) : v \in \tilde{\mathcal{E}}(\mathbb{R}^2),\ p(v) = \mathfrak{p} \big\}.$$

However, similar to (5.1.4), we have that

$$\inf \big\{ E(v) : v \in \tilde{\mathcal{E}}(\mathbb{R}^2), \quad d(v) \neq 0 \big\} = 4\pi,$$

which shows that high energy solutions cannot be obtained by considering functions with nonzero degree. For these reasons, we consider the minimizing curve

$$E^0_{\min}(\mathfrak{p}) = \inf \big\{ E(v) : v \in \tilde{\mathcal{E}}(\mathbb{R}^2),\ p(v) = \mathfrak{p},\ d(v) = 0 \big\},$$

for which we establish the following results.

Theorem 5.1.1. *The function $\mathfrak{p} \to E^0_{\min}(\mathfrak{p})$ is concave, nondecreasing and Lipschitz continuous. Moreover,*

$$|E^0_{\min}(\mathfrak{p}) - E^0_{\min}(\mathfrak{q})| \leq |\mathfrak{p} - \mathfrak{q}|,$$

for all $\mathfrak{p}, \mathfrak{q} > 0$. In particular

$$E^0_{\min}(\mathfrak{p}) \leq \mathfrak{p}, \quad \textit{for all } \mathfrak{p} > 0,$$

and the map $\Xi(\mathfrak{p}) := \mathfrak{p} \to \mathfrak{p} - E^0_{\min}(\mathfrak{p})$ is continuous, convex and nondecreasing on $\mathbb{R}_+$. In particular, there exists $\mathfrak{p}_0 \geq 0$ such that $\Xi(\mathfrak{p}) = 0$, for all $\mathfrak{p} \leq \mathfrak{p}_0$.

Remark 5.1.2. If the constant $\mathfrak{p}_0$ given by Theorem 5.1.1 is positive, it can be shown that for any $\mathfrak{p} \in (0, \mathfrak{p}_0)$, the infimum of $E^0_{\min}(\mathfrak{p})$ is not achieved.

As a consequence of Theorem 5.1.1, the lateral derivatives of $E^0_{\min}$ exist for any $\mathfrak{p} > 0$ and satisfy

$$0 \leq \frac{d^+}{d\mathfrak{p}}\big(E^0_{\min}(\mathfrak{p})\big) \leq \frac{d^-}{d\mathfrak{p}}\big(E^0_{\min}(\mathfrak{p})\big) \leq 1. \tag{5.1.11}$$

Moreover, the lateral derivatives coincide except for a (possibly empty) countable set.

Proposition 5.1.3. *Let $\mathfrak{p} > 0$ and assume that $E^0_{\min}(\mathfrak{p})$ is attained by a function $u_\mathfrak{p}$. Then $u_\mathfrak{p}$ is a smooth solution of (TW$_c$) of speed $c = c(u_\mathfrak{p})$ satisfying*

$$0 \leq \frac{d^+}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \leq c(u_\mathfrak{p}) \leq \frac{d^-}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \leq 1. \tag{5.1.12}$$

Moreover $c(u_\mathfrak{p}) \in (0, 1)$.

Remark 5.1.4. It is also possible to prove that, up to a translation, that $u_\mathfrak{p}$ is axisymmetric, i.e. there exists a function $\mathfrak{u}_\mathfrak{p} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$u_\mathfrak{p}(x) = \mathfrak{u}_\mathfrak{p}(x_1, |x_2|), \text{ for all } x = (x_1, x_2) \in \mathbb{R}^2.$$

Theorem 5.1.1 and Proposition 5.1.3 are in agreement with the results in [83] and Figure 5.1. Thus these results could help to establish the existence of traveling waves by variational methods. One of our main results has as consequence that the minimizing curve does not reach its infimum close to $\mathfrak{p} = 0$ (see Theorem 5.1.10). This justifies that the curve of solutions in Figure 5.1 is far from the origin, a very important difference compared to the solutions of the Gross–Pitaevskii equation in dimension two.

5.1.3 Remarks on regularity of traveling waves

In the study of general properties of the equation (TW_c), the first problem is the regularity of finite energy solutions. We notice that the square-gradient term prevents us from invoking the usual elliptic regularity estimates. Moreover, a well-known regularity result says that every *continuous* solution in $H^1(\Omega)$ of such an elliptic system with quadratic growth belongs to $H^{2,2}_{\mathrm{loc}}(\Omega) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ (see [42, 69, 16, 64]). In dimension $N = 2$ some elements of the theory of harmonic maps (see e.g. [58]) can be adapted to prove the continuity of solutions in $\mathcal{E}(\mathbb{R}^2)$, leading to the following result.

Proposition 5.1.5. *Let $c \geq 0$ and $u \in \mathcal{E}(\mathbb{R}^2)$ be a solution of* (TW_c). *Then $u \in C^\infty(\mathbb{R}^2) \cap \tilde{\mathcal{E}}(\mathbb{R}^2)$, $u_3 \in L^p(\mathbb{R}^2)$ for all $p \in [2, \infty]$ and $\nabla u \in W^{k,p}(\mathbb{R}^2)$ for all $k \in \mathbb{N}$ and $p \in [2, \infty]$. Moreover, there exist constants $\varepsilon_0 > 0$ and $K(\varepsilon_0) > 0$ such that*

$$\|u_3\|_{L^\infty(\mathbb{R}^2)} \leq K(\varepsilon_0)(1 + c)E(u)^{1/2}, \tag{5.1.13}$$

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq K(\varepsilon_0)(1 + c)E(u)^{1/4}, \tag{5.1.14}$$

provided that $E(u) \leq \varepsilon_0$.

On the other hand, in dimension $N \geq 3$, it is not possible to have such a result. Indeed, if we consider the equation

$$-\Delta v = |\nabla v|^2 v, \quad \text{in } B(0, 1) \subset \mathbb{R}^N, \ v \in \mathbb{S}^2, \tag{5.1.15}$$

we have that $v(x) = x \to x/|x|$ is a finite energy solution discontinuous at the origin. Moreover T. Rivière [90] proved that (5.1.15) has discontinuous solutions almost everywhere, for any $N \geq 3$. However, the solutions found in [74] and [83] of (TW_c) are smooth. Thus, in higher dimensions, we will assume that the traveling waves belong to the space $\mathcal{E}(\mathbb{R}^N) \cap UC(\mathbb{R}^N)$, where $UC(\mathbb{R}^N)$ denotes the set of uniformly continuous functions. Under this assumption, by classical arguments (see e.g. [16, 69, 64, 80]) we can deduce that the solutions of our problem are smooth and we can establish the following estimates.

Lemma 5.1.6. *Let $N \geq 3$, $c \geq 0$ and $u \in \mathcal{E}(\mathbb{R}^N) \cap UC(\mathbb{R}^N)$ be a solution of* (TW_c). *Then $u \in C^\infty(\mathbb{R}^N) \cap \tilde{\mathcal{E}}(\mathbb{R}^N)$ and $\nabla u \in W^{k,p}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$ and $p \in [2, \infty]$. Moreover, if $N \in \{3, 4\}$ and $c \in [0, 1]$, there exist $\varepsilon_0, K, \alpha > 0$, independent of u and c, such that*

$$\|u_3\|_{L^\infty(\mathbb{R}^N)} \leq KE(u)^\alpha \tag{5.1.16}$$

and

$$\|\nabla u\|_{L^\infty(\mathbb{R}^N)} \leq KE(u)^\alpha, \tag{5.1.17}$$

provided that $E(u) \leq \varepsilon_0$.

5.1.4 Nonexistence results and asymptotic behavior at infinity

Our main result is in the same spirit of a result proved by the author for the Gross–Pitaevskii equation in [30] (see also [8]). Precisely, we show the existence of a lower bound for the energy, which implies the nonexistence of nontrivial traveling waves of small energy.

Theorem 5.1.7. *Let $N \in \{3,4\}$. Then there exists a constant $\mu > 0$ such that for any $u \in \mathcal{E}(\mathbb{R}^N) \cap UC(\mathbb{R}^N)$ nontrivial solution of* (TW_c) *with $c \in (0,1]$, we have*

$$E(u) \geq \mu. \tag{5.1.18}$$

In particular, nontrivial traveling waves of small energy with speed $c \in (0,1]$ do not exist.

As noticed in [56] in dimension two, there is no smooth static solution of (TW_c), i.e. with speed $c = 0$. More generally, we obtain the following result for static waves.

Proposition 5.1.8. *Let $N \geq 2$. Assume that $u \in \mathcal{E}(\mathbb{R}^N)$ is a solution of* (TW_c) *with $c = 0$. Suppose also that $u \in UC(\mathbb{R}^N)$ if $N \geq 3$. Then u is a trivial solution.*

In the case of dimension $N = 2$, the nonexistence problem is more delicate and we need to suppose that the energy is controlled by the momentum.

Theorem 5.1.9. *For every $M \geq 0$, there exists a constant $\kappa_M > 0$ such that for any $u \in \mathcal{E}(\mathbb{R}^2)$ nontrivial solution of* (TW_c) *with $c \in (0,1)$, we have*

$$E(u) \geq \kappa_M, \tag{5.1.19}$$

provided that $E(u) \leq p(u) + M(1-c^2)$. In particular, taking $M = 0$, nontrivial traveling waves with speed $c \in (0,1)$ such that their energy is less than or equal to their momentum do not exist.

Although the condition $E(u) \leq p(u)$ restricts the set of traveling waves, it is sufficient to establish the nonexistence of minimizing solutions given by the curve $E^0_{\min}(\mathfrak{p})$.

Theorem 5.1.10. *Let κ_0 be the constant given by Theorem 5.1.9 with $M = 0$. Then for any $0 < \mathfrak{p} < \kappa_0$ the infimum of the minimization problem associated to $E^0_{\min}(\mathfrak{p})$ is not achieved.*

Even though we cannot show that the condition $E(u) \leq p(u) + M(1-c^2)$ is valid in general, we can prove that a similar a priori estimate holds.

Proposition 5.1.11. *Let $c \in (0,1)$. Assume that $u \in \mathcal{E}(\mathbb{R}^2)$ is a nontrivial solution of* (TW_c). *Then there exists $\varepsilon_0 > 0$ such that*

$$E(u) \leq p(u) + \frac{31}{6}\|u_3\|^2_{L^\infty(\mathbb{R}^2)} p(u),$$

provided that $E(u) \leq \varepsilon_0$. In particular, for all $L > 1$, there exists $\varepsilon(L) > 0$, independent of u and c, such that

$$E(u) \leq L p(u),$$

provided that $E(u) \leq \varepsilon(L)$.

Proposition 5.1.11 shows that if one can prove an a priori estimate such as

$$\|u_3\|^2_{L^\infty(\mathbb{R}^2)} p(u) \leq M(1-c^2),$$

for some universal constant M, then Theorem 5.1.9 holds without the restriction $E(u) \leq p(u) + M(1-c^2)$.

A key point in our study of (TW_c) is that the lifting (5.1.7) allows us to obtain the equation

$$\Delta^2 u_3 - \Delta u_3 + c^2 \partial_{11}^2 u_3 = -\Delta F + c\partial_1(\operatorname{div} G), \quad \text{ on } \mathbb{R}^N, \tag{5.1.20}$$

where

$$G = (G_1, G_2) := u_1 \nabla u_2 - u_2 \nabla u_1 - \nabla(\chi\theta), \quad \text{ on } \mathbb{R}^N,$$

$F = 2e(u)u_3 + cG_1$, and χ is a $C^\infty(\mathbb{R}^N)$-function such that $|\chi| \leq 1$, $\chi = 0$ on $B(0, 2R)$ and $\chi = 1$ on $B(0, 3R)^c$, if $R > 0$. In the case that $R = 0$, it is enough to take $\chi = 1$ on $\mathbb{R}^N$. We note that the differential operator

$$\Delta^2 - \Delta + c^2 \partial_{11}^2$$

is elliptic if and only if $c \leq 1$, which shows that $c = 1$ is a critical value for the equation (TW_c).

From (5.1.20) we can obtain the convolution equation

$$u_3 = \mathcal{L}_c * F - c\sum_{j=1}^{N} \mathcal{L}_{c,j} * G_j, \tag{5.1.21}$$

where

$$\widehat{\mathcal{L}}_c = \frac{|\xi|^2}{|\xi|^4 + |\xi|^2 - c^2\xi_1^2}, \quad \widehat{\mathcal{L}}_{c,j} = \frac{\xi_1\xi_j}{|\xi|^4 + |\xi|^2 - c^2\xi_1^2}, \tag{5.1.22}$$

and a similar identity holds for $\nabla(\chi\theta)$. Using these equations, arguments introduced in [15, 28, 48, 51] allow us to establish the decay of the solutions for $c \in (0, 1)$. Moreover, since the kernels in (5.1.22) are the same as those appearing in the Gross–Pitaevskii equation, after proving some algebraic decay of the solutions of (TW_c) (see Corollary 5.7.2), we can apply the theory developed in [48] to obtain the following estimates.

Proposition 5.1.12. *Let $N \geq 2$ and $c \in (0, 1)$. Assume that $u \in \mathcal{E}(\mathbb{R}^N)$ is a solution of* (TW_c). *Suppose further that $u \in UC(\mathbb{R}^N)$ if $N \geq 3$. Then there exist constants $R(u), K(c, u) \geq 0$ such that*

$$|u_3(x)| + |\nabla\theta(x)| + |\nabla\check{u}(x)| \leq \frac{K(c,u)}{1 + |x|^N}, \tag{5.1.23}$$

$$|\nabla u_3(x)| + |D^2\theta(x)| + |D^2\check{u}(x)| \leq \frac{K(c,u)}{1 + |x|^{N+1}}, \tag{5.1.24}$$

$$|D^2 u_3(x)| \leq \frac{K(c,u)}{1 + |x|^{N+2}}, \tag{5.1.25}$$

for all $x \in B(0, R(u))^c$.

Finally, using Proposition 5.1.12 and the arguments in [51], we can compute precisely the limit at infinity of these solutions.

Theorem 5.1.13. *Let $N \geq 2$ and $c \in (0,1)$. Assume that $u \in \mathcal{E}(\mathbb{R}^N)$ is a solution of* (TW_c). *Suppose further that $u \in UC(\mathbb{R}^N)$ if $N \geq 3$. Then there exist a constant $\lambda_\infty \in \mathbb{C}$ and two functions $\check{u}_\infty, u_{3,\infty} \in C(\mathbb{S}^{N-1};\mathbb{R})$ such that*

$$|x|^{N-1}(\check{u}(x) - \lambda_\infty) - i\lambda_\infty \check{u}_\infty \left(\frac{x}{|x|}\right) \to 0, \tag{5.1.26}$$

$$|x|^N u_3(x) - u_{3,\infty} \left(\frac{x}{|x|}\right) \to 0, \tag{5.1.27}$$

uniformly as $|x| \to \infty$. Moreover, assuming without loss of generality that $\lambda_\infty = 1$, we have

$$\check{u}_\infty(\sigma) = \frac{\alpha\sigma_1}{(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}} + \sum_{j=2}^{N} \frac{\beta_j\sigma_j}{(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}}, \tag{5.1.28}$$

$$u_{3,\infty}(\sigma) = \alpha c \left(\frac{1}{(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}} - \frac{N\sigma_1^2}{(1-c^2+c^2\sigma_1^2)^{\frac{N+2}{2}}} \right) - Nc \sum_{j=2}^{N} \beta_j \frac{\sigma_1\sigma_j}{(1-c^2+c^2\sigma_1^2)^{\frac{N+2}{2}}}, \tag{5.1.29}$$

where $\sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{S}^{N-1}$,

$$\alpha = \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}(1-c^2)^{\frac{N-3}{2}} \left(2c \int_{\mathbb{R}^N} e(u)u_3\,dx - (1-c^2)\int_{\mathbb{R}^N} G_1(x)\,dx \right)$$

and

$$\beta_j = -\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}(1-c^2)^{\frac{N-1}{2}} \int_{\mathbb{R}^N} G_j(x)\,dx.$$

Notations. For a function m taking values on $\mathbb{R}^3$, that is $m = (m_1, m_2, m_3)$, we define $\check{m}$ as the complex-valued function $\check{m} = m_1 + im_2$. The N-dimensional unit sphere is $\mathbb{S}^N = \{x \in \mathbb{R}^{N+1} : |x| = 1\}$ and in the case $N = 1$ we also use the identification $\mathbb{S}^1 \cong \{z \in \mathbb{C} : |z| = 1\}$. The inner product in $\mathbb{R}^3$ will be denoted by $\langle \cdot, \cdot \rangle$ or just $\cdot$ and the cross product by $\times$. Given $x = (x_1, x_2)$, $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f = (f_1, f_2)$, $g, h : \mathbb{R}^2 \to \mathbb{R}^3$, $g = (g_1, g_2, g_3)$, $h = (h_1, h_2, h_3)$, we set $x^\perp = (-x_2, x_1)$, $\nabla^\perp = (-\partial_2, \partial_1)$, $\mathrm{curl}(f) = \partial_1 f_2 - \partial_2 f_1$ and

$$\nabla g : \nabla h = \sum_{i=1}^{3} \nabla g_i \cdot \nabla h_i.$$

Let $A \in \mathbb{R}^{3\times 3}$ and $b \in \mathbb{R}^3$, then the product of A and b is denoted by $A.b \in \mathbb{R}^3$. For $y \in \mathbb{R}^N$ and $r \geq 0$, $B(y, r)$ or $B_r(y)$ denote the open ball of center y and radius r (which is empty for $r = 0$). In the case that there is no confusion, we simply put B_r.

$\mathcal{F}(f)$ or $\widehat{f}$ stand for the Fourier transform of f, namely

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^N} f(x)e^{-ix\cdot\xi}\,dx.$$

We also adopt the standard notation $K(\cdot, \cdot, \dots)$ to represent a generic constant that depends only on each of its arguments.

5.2 Regularity of traveling waves

In this section we use some of the elements developed to the study of the harmonic map equation. In particular, the next lemma is a consequence of the Wente lemma [102, 21, 100] and Hélein's trick [57, 58].

Lemma 5.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $g \in L^2(\Omega)$. Assume that $u \in H^1(\Omega, \mathbb{S}^2)$ satisfies*

$$-\Delta u = |\nabla u|^2 u + g, \qquad \text{in } \Omega. \tag{5.2.1}$$

Let $r > 0$ and $x \in \Omega$ such that $B(x,r) \subseteq \Omega$. Then for any $i \in \{1,2,3\}$ we have

$$\begin{aligned}\operatorname*{osc}_{B(x,r/2)} u_i \leq K\Big(&\min\big\{\|\nabla u\|_{L^2(B(x,r))}, \|u_i\|_{L^\infty(\partial B_r)}\big\} + \|\nabla u\|^2_{L^2(B(x,r))} \\ &+ r\|g\|_{L^2(B(x,r))}\big(1 + \|\nabla u\|_{L^2(B(x,r))}\big)\Big),\end{aligned} \tag{5.2.2}$$

for some universal constant $K > 0$. In particular $u \in C(\Omega)$. Moreover, if the trace of u on $\partial\Omega$ belongs to $C(\partial\Omega)$, then $u \in C(\bar{\Omega})$ and

$$\|u_i\|_{L^\infty(\Omega)} \leq \|u_i\|_{L^\infty(\partial\Omega)} + K(\Omega)(\|\nabla u\|^2_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}(1 + \|\nabla u\|_{L^2(\Omega)})), \tag{5.2.3}$$

for some constant $K(\Omega)$ depending only on Ω. Furthermore, when $\Omega = \omega_R \equiv \{Rx,\ x \in \omega\}$, where ω is a fixed domain and $R > 0$, estimate (5.2.3) *can be written as*

$$\|g_i\|_{L^\infty(\omega_R)} \leq \|g_i\|_{L^\infty(\partial\omega_R)} + K\Big(\|\nabla g\|^2_{L^2(\omega_R))} + R\|f\|_{L^2(\omega_R)}\big(1 + \|\nabla g\|_{L^2(\omega_R)}\big)\Big), \tag{5.2.4}$$

where K only depends on ω.

Proof. As for the standard harmonic maps, we recast (5.2.1) as

$$-\Delta u_i = \sum_{j=1}^{3} v_{ij} \cdot \nabla u_j + g_i, \quad \text{in } \Omega, \quad i = 1,2,3, \tag{5.2.5}$$

where $v_{i,j} = u_i \nabla u_j - u_j \nabla u_i$. Then

$$\operatorname{div}(v_{i,j}) = u_j g_i - u_i g_j \quad \text{ and } \quad \|\operatorname{div}(v_{i,j})\|_{L^2(\Omega)} \leq 2\|g\|_{L^2(\Omega)}. \tag{5.2.6}$$

Let us consider $h_{i,j} \in H^2(\Omega)$ the solution of

$$\begin{cases} \Delta h_{i,j} = \operatorname{div}(v_{i,j}), & \text{in } \Omega, \\ h_{i,j} = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.2.7}$$

Thus

$$\|\nabla h_{i,j}\|_{L^2(\Omega)} \leq \|v_{i,j}\|_{L^2(\Omega)} \leq 2\|\nabla u\|_{L^2(\Omega)}. \tag{5.2.8}$$

Since $\operatorname{div}(v_{i,j} - \nabla h_{i,j}) = 0$, with $v_{i,j} - \nabla h_{i,j} \in L^2(\Omega)$, there exists $w_{i,j} \in H^1(\Omega)$ (see e.g. [44, Thm 2.9]) such that

$$v_{i,j} = \nabla h_{i,j} + \nabla^\perp w_{ij}, \quad \text{in } \Omega. \tag{5.2.9}$$

Now we decompose u as $u_i = \phi_i + \varphi_i + \psi_i$, where $\phi_i, \varphi_i, \psi_i$ are the solutions of the equations

$$\begin{cases} -\Delta\phi_i = 0 & \text{in } U, \\ \quad \phi_i = u_i, & \text{on } \partial U, \end{cases} \tag{5.2.10}$$

$$\begin{cases} -\Delta\varphi_i = \nabla h \cdot \nabla u + g_i, & \text{in } U, \\ \quad \varphi_i = 0, & \text{on } \partial U, \end{cases} \tag{5.2.11}$$

$$\begin{cases} -\Delta\psi_i = \nabla^\perp w \cdot \nabla u, & \text{in } U, \\ \quad \psi_i = 0, & \text{on } \partial U, \end{cases} \tag{5.2.12}$$

where U is an open smooth domain such that $B_r \subseteq U \subseteq \Omega$. We now prove (5.2.2) for $r = 1$, supposing that $B_1 \subseteq U$, since then (5.2.2) follows from a scaling argument. First, invoking Theorem B.1 we have that

$$\operatorname*{osc}_{B_{1/2}} \phi_i \leq K \min\left\{\|\nabla\phi_i\|_{L^2(B_{2/3})}, \|\phi_i\|_{L^2(B_{2/3})}\right\}. \tag{5.2.13}$$

Also, some standard computations and the maximum principle yield

$$\|\nabla\phi_i\|_{L^2(U)} \leq \|\nabla u_i\|_{L^2(U)} \quad \text{and} \quad \|\phi_i\|_{L^\infty(U)} \leq \|u_i\|_{L^\infty(\partial U)}. \tag{5.2.14}$$

Thus from (5.2.13) and (5.2.14) we conclude that

$$\operatorname*{osc}_{B_{1/2}} \phi_i \leq K \min\left\{\|\nabla u_i\|_{L^2(U)}, \|u_i\|_{L^\infty(\partial U)}\right\}. \tag{5.2.15}$$

For φ_i, Theorem B.2 gives

$$\operatorname*{osc}_{B_1} \varphi_i \leq K(\|\nabla h \cdot \nabla u\|_{L^{3/2}(B_1)} + \|g\|_{L^2(B_1)}). \tag{5.2.16}$$

To estimate the first term in the r.h.s. of (5.2.16), we use the Hölder inequality

$$\|\nabla h \cdot \nabla u\|_{L^{3/2}(B_1)} \leq \|\nabla h\|_{L^6(B_1)} \|\nabla u\|_{L^2(B_1)}, \tag{5.2.17}$$

and the Sobolev embedding theorem

$$\|\nabla h\|_{L^6(B_1)} \leq K(\|\nabla h\|_{L^2(B_1)} + \|D^2 h\|_{L^2(B_1)}). \tag{5.2.18}$$

By combining (5.2.6), (5.2.16), (5.2.17), (5.2.18) and L^2-regularity estimates for (5.2.7), we are led to

$$\operatorname*{osc}_{B_1} \varphi_i \leq K\|g\|_{L^2(B_1)}(1 + \|\nabla u\|_{L^2(B_1)}). \tag{5.2.19}$$

Similarly, since $W^{2,p}(U) \hookrightarrow C(\bar{U})$, for all $p > 1$, we also have

$$\|\varphi_i\|_{C(\bar{U})} \leq K(U)\|g\|_{L^2(U)}(1 + \|\nabla u\|_{L^2(U)}). \tag{5.2.20}$$

To estimate ψ_i we invoke the Wente estimate (see [100], [58]), so that

$$\|\psi_i\|_{C(\bar{U})} + \operatorname*{osc}_{U} \psi_i \leq K\|\nabla w\|_{L^2(U)}\|\nabla u\|_{L^2(U)} \leq K\|\nabla u\|^2_{L^2(U)}, \tag{5.2.21}$$

where we have used (5.2.8) and (5.2.9) for the last inequality.

Therefore, taking $U = B_1$ and putting together (5.2.15), (5.2.19) and (5.2.21), we conclude (5.2.2) with $r = 1$.

If the trace of u on $\partial\Omega$ belongs to $C(\partial\Omega)$, we take $\Omega = U$ and then from (5.2.10) we have that $\phi_i \in C^2(\Omega) \cap C(\bar{\Omega})$. Since $\varphi_i, \psi_i \in C(\bar{\Omega})$, we conclude that $u_i \in C(\bar{\Omega})$ and (5.2.3) follows from (5.2.14), (5.2.20) and (5.2.21). Finally, using again a scaling argument, we get (5.2.4). □

Let us consider the elliptic system,

$$\Delta u = f(x, u, \nabla u), \quad \text{in } \Omega,$$

where Ω is a smooth open bounded set and f is a smooth function with quadratic growth

$$|f(x, z, p)| \leq A + B|p|^2.$$

As mentioned before, well-known regularity results imply that every *continuous* solution in $H^1(\Omega)$ belongs to $H^{2,2}_{\text{loc}}(\Omega) \cap C^{0,\alpha}_{loc}(\Omega)$. Using some of the arguments that lead to this result, we get the following estimate for $\|\nabla u\|_{L^\infty}$.

Lemma 5.2.2. *Let $y \in \mathbb{R}^2$, $r > 0$ and $B_r \equiv B(y, r)$. Assume that $u \in H^1(B_r, \mathbb{S}^2)$ satisfies*

$$-\Delta u = |\nabla u|^2 u + f(x, u(x), \nabla u(x)), \text{ in } B_r, \tag{5.2.22}$$

where $f \in L^\infty(B_r) \times C(\mathbb{R}^3) \times C(\mathbb{R}^{3\times 3})$ and $|f(x, z, p)| \leq C_1 + C_2|p|$, for some constants $C_1, C_2 \geq 0$, for all $x \in B_r$, $z \in \mathbb{R}^3$, $p \in \mathbb{R}^{3\times 3}$. Suppose that

$$A \equiv A(u, r) := \frac{\operatorname{osc}_{B_r} u(1 + r^2(C_1 + C_2^2))}{1 - 3 \operatorname{osc}_{B_r} u} \leq \frac{1}{32}. \tag{5.2.23}$$

Then

$$\|D^2 u\|_{L^2(B_{r/2})} + \|\nabla u\|^2_{L^4(B_{r/2})} \leq K r^{-1} \left(\|\nabla u\|_{L^2(B_r)} + \|g\|_{L^2(B_r)}\right), \tag{5.2.24}$$

where $g(x) = f(x, u(x), \nabla u(x))$. Assume further that $f(x, z, p) = \tilde{f}(x) + R_f(x, z, p)$, for some functions $\tilde{f} \in L^\infty(B_r)$, $R_f \in L^\infty(B_r) \times C(\mathbb{R}^3) \times C(\mathbb{R}^{3\times 3})$, with $|R_f(x, z, p)| \leq C_3|p|$, for some constant $C_3 \geq 0$, for all $x \in B_r$, $z \in \mathbb{R}^3$, $p \in \mathbb{R}^{3\times 3}$. Then,

$$\begin{aligned}\|\nabla u\|_{L^\infty(B_{r/4})} \leq & K r^{-1} \|\nabla u\|_{L^2(B_r)} + K r^{-2/3} \Big(\|\nabla u\|^2_{L^2(B_r)}(r^{-2} + r^{-4/3}) + \|g\|^2_{L^2(B_r)} \\ & + \|\tilde{f}\|_{L^3(B_r)} + C_3 r^{-1/3} \|\nabla u\|^{1/3}_{L^2(B_r)} \big(\|\nabla u\|^{1/3}_{L^2(B_r)} + \|g\|^{1/3}_{L^2(B_r)}\big)\Big),\end{aligned} \tag{5.2.25}$$

where K is some universal constant.

Proof. As mentioned before, Lemma 5.2.1 and the quadratic growth of the r.h.s. of (5.2.22) imply that $u \in H^{2,2}_{\text{loc}}(\Omega)$. In fact, this could be seen by repeating the following arguments with finite differences instead of weak derivatives. As standard in the analysis of this type of equations, we let $\rho \in (0, r)$ and $\chi \in C_0^\infty(B_r)$, with $\chi(x) = 1$ if $|x| \leq \rho$,

$$|\chi| \leq 1 \quad \text{and} \quad |\nabla \chi| \leq K/(r - \rho), \quad \text{on } B_r. \tag{5.2.26}$$

Then setting $\eta = \chi|\nabla u|$, taking inner product in (5.2.1) with $(u - u(x_0))\eta^2$ and integrating by parts we obtain

$$\int_{B_r} |\nabla u|^2 \eta^2 + 2\int_{B_r} \langle \nabla u.(u - u(x_0)), \eta \nabla \eta \rangle = \int_{B_r} |\nabla u|^2 u \cdot (u - u(x_0))\eta^2 + \int_{B_r} \eta^2 g \cdot (u - u(x_0)). \tag{5.2.27}$$

Then, using the elementary inequality $2ab \leq a^2 + b^2$,

$$|\eta^2 g \cdot (u - u(x_0))| \leq \eta^2 (C_1 + C_2 |\nabla u|) \operatorname*{osc}_{B_r}(u) \leq C_1 \eta^2 \operatorname*{osc}_{B_r}(u) + \eta^2 |\nabla u|^2 \operatorname*{osc}_{B_r}(u) + \frac{1}{4} C_2^2 \eta^2 \operatorname*{osc}_{B_r}(u).$$

In a similar fashion, we estimate the remaining terms in (5.2.27). Then, using the Poincaré inequality

$$\|\eta\|_{L^2(B_r)} \leq \frac{r}{j_0}\|\nabla\eta\|_{L^2(B_r)},$$

where $j_0 \approx 2.4048$ is the first zero of the Bessel function, and that $|u| = 1$, we conclude that

$$\int_{B_r} |\nabla u|^2\eta^2 \leq \frac{\operatorname{osc}_{B_r} u(1 + r^2(C_1 + C_2^2))}{1 - 3\operatorname{osc}_{B_r} u}\int_{B_r} |\nabla\eta|^2,$$

where we bounded $1/j_0$ and $1/(4j_0)$ by 1 to simplify the estimate. Thus,

$$\int_{B_r} |\nabla u|^4\chi^2 \leq A\int_{B_r}(|D^2u|^2\chi^2 + |\nabla u|^2|\nabla\chi|^2)). \tag{5.2.28}$$

On the other hand, taking inner product in (5.2.22) with $\partial_k(\chi^2\partial_k u)$, integrating by parts and summing over $k = 1, 2$, we have

$$-\int_{B_r}\chi^2|D^2u|^2 - 2\sum_{\substack{i\in\{1,2\}\\ j,k\in\{1,2,3\}}}\int_{B_r}\partial_{jk}u_i\chi\partial_j\chi\partial_k u_i = \sum_{i\in\{1,2\}}\int_{B_r}(|\nabla u|^2u_i + g_i)(2\chi\nabla\chi\nabla u_i + \chi^2\Delta u_i).$$

Using again the inequalities $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ and $ab \leq \varepsilon a^2 + b^2/4\varepsilon$, we are led to

$$\int_{B_r}\chi^2|D^2u|^2 \leq \frac{1}{1-3\varepsilon}\int_{B_r}\left((2+\varepsilon^{-1})|\nabla u|^2|\nabla\chi|^2 + (1+4\varepsilon^{-1})|\nabla u|^4\chi^2 + (1+4\varepsilon^{-1})\chi^2|g^2|\right).$$

Then, minimizing with respect to ε, it follows that

$$\int_{B_r}\chi^2|D^2u|^2 \leq 16\int_{B_r}(|\nabla u|^2|\nabla\chi|^2 + |\nabla u|^4\chi^2 + \chi^2|g^2|). \tag{5.2.29}$$

Combining (5.2.26), (5.2.28) and (5.2.29), we infer that

$$\int_{B_\rho}|\nabla u|^4 \leq \frac{KA}{1-16A}\left(\frac{1}{(r-\rho)^2}\int_{B_r}|\nabla u|^2 + \int_{B_r}|g|^2\right), \tag{5.2.30}$$

$$\int_{B_\rho}|D^2u|^2 \leq \frac{K}{1-16A}\left(\frac{1+A}{(r-\rho)^2}\int_{B_r}|\nabla u|^2 + \int_{B_r}|g|^2\right). \tag{5.2.31}$$

Taking $\rho = r/2$ and using that $A \leq 1/32$, (5.2.24) follows.

Now we decompose u as $u_i = \phi_i + \psi_i$, where

$$\begin{cases} -\Delta\phi_i = 0, & \text{in } B_{r/2},\\ \phi_i = u_i, & \text{on } \partial B_{r/2}. \end{cases} \tag{5.2.32}$$

$$\begin{cases} -\Delta\psi_i = |\nabla u|^2u_i + \tilde{f}_i + (R_f(x,u,\nabla u))_i, & \text{in } B_{r/2},\\ \psi_i = 0, & \text{on } \partial B_{r/2}, \end{cases} \tag{5.2.33}$$

Since ϕ_i is a harmonic function,

$$\|\nabla\phi_i\|_{L^\infty(B_{r/4})} \leq Kr^{-1}\|\nabla\phi_i\|_{L^2(B_{r/2})},$$

so that using also (5.2.14), we obtain the estimate

$$\|\nabla \phi_i\|_{L^\infty(B_{r/4})} \leq K r^{-1} \|\nabla u_i\|_{L^2(B_{r/2})}. \tag{5.2.34}$$

For ψ_i, we recall that using the L^p-regularity theory for the Laplacian and a scaling argument, the solution $v \in H_0^1(B_R)$ of the equation $-\Delta v = h$ satisfies

$$\|\nabla v\|_{L^\infty(B_R)} \leq K(p) R^{1-2/p} \|h\|_{L^p(B_R)}, \quad \text{for all } p > 2.$$

Applying this estimate with $p = 3$ to (5.2.33), we get

$$\|\nabla \psi_i\|_{L^\infty(B_{r/2})} \leq C r^{-2/3} \left(\|\nabla u\|^2_{L^6(B_{r/2})} + \|\tilde{f}\|_{L^3(B_{r/2})} + C_3 \|\nabla u\|_{L^3(B_{r/2})} \right). \tag{5.2.35}$$

Also, by the Sobolev embedding theorem, we have

$$\|\nabla u\|_{L^6(B_{r/2})} \leq K \left(\|D^2 u\|_{L^2(B_{r/2})} + r^{-2/3} \|\nabla u\|_{L^2(B_{r/2})} \right). \tag{5.2.36}$$

Therefore, by putting together (5.2.24), (5.2.34), (5.2.35), (5.2.36) and the interpolation inequality

$$\|\nabla u\|_{L^3(B_{r/2})} \leq \|\nabla u\|^{1/3}_{L^2(B_{r/2})} \|\nabla u\|^{2/3}_{L^4(B_{r/2})},$$

we deduce (5.2.25). □

Now we turn back to equation (TW_c). By setting

$$E_{x,r}(u) = \int_{B(x,r)} e(u), \quad x \in \mathbb{R}^2, r > 0,$$

we obtain the following result.

Corollary 5.2.3. *There exist $\varepsilon_0 > 0$ and a positive constant $K(\varepsilon_0)$, such that for any $c \geq 0$ and $u \in \mathcal{E}(\mathbb{R}^2)$ solution of (TW_c) satisfying*

$$E_{x,r}(u) \leq \varepsilon_0,$$

for some $x \in \mathbb{R}^2$ and $r \in (0, 1]$, we have

$$\operatorname*{osc}_{B(x,r/2)} u \leq K(1+c) E_{x,r}(u)^{1/2}, \tag{5.2.37}$$

$$\|\nabla u\|_{L^\infty(B(x,r/4))} \leq K(1+c) E_{x,r}(u)^{1/4} r^{-2/3}. \tag{5.2.38}$$

In particular, if $E(u) \leq \varepsilon_0$, then

$$\|u_3\|_{L^\infty(\mathbb{R}^2)} \leq K(1+c) E(u)^{1/2}, \tag{5.2.39}$$

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq K(1+c) E(u)^{1/4}. \tag{5.2.40}$$

Proof. Estimates (5.2.37) and (5.2.38) follow from Lemmas 5.2.1 and 5.2.2. Then, taking $r = 1$, we conclude that (5.2.40) holds. Now we turn to (5.2.39). For any $y \in \mathbb{R}^2$ we have

$$2E(u) \geq \int_{B(y,1/2)} u_3^2 \geq \frac{\pi}{4} \min_{B(y,1/2)} |u_3|^2. \tag{5.2.41}$$

On the other hand, by Lemma 5.2.1,

$$\max_{B(y,1/2)} |u_3| \leq \operatorname*{osc}_{B(y,1/2)} u_3 + \min_{B(y,1/2)} |u_3| \leq K(1+c) E(u)^{1/2} + \min_{B(y,1/2)} |u_3|. \tag{5.2.42}$$

By combining (5.2.41) and (5.2.42), we are led to (5.2.39). □

Proof of Proposition 5.1.5. Since u has finite energy, for every $\varepsilon > 0$, there exists $\rho > 0$ such that for all $y \in \mathbb{R}^2$

$$E_{y,\rho}(u) \leq \varepsilon. \tag{5.2.43}$$

In fact, since $e(u) \in L^1(\mathbb{R}^2)$, by Lemma B.3, for every $\varepsilon > 0$ we can decompose $e(u) = e_{1,\varepsilon}(u) + e_{2,\varepsilon}(u)$ such that

$$\|e_{1,\varepsilon}(u)\|_{L^1(\mathbb{R}^2)} \leq \varepsilon/2 \quad \text{ and } \quad \|e_{2,\varepsilon}(u)\|_{L^\infty(\mathbb{R}^2)} \leq K_\varepsilon,$$

for some constant K_ε. Then for any $y \in \mathbb{R}^2$,

$$\|e_{2,\varepsilon}(u)\|_{L^1(B(y,\rho))} \leq K_\varepsilon \pi \rho^2.$$

Taking

$$\rho = \left(\frac{\varepsilon}{2K_\varepsilon \pi}\right)^{1/2},$$

we obtain (5.2.43). Thus, invoking Corollary 5.2.1, with $\varepsilon = \varepsilon_0$ and $r = \min\{1, \rho\}$, we conclude that

$$\|\nabla u\|_{L^\infty(B(y,r/2))} \leq K(1+c)\varepsilon_0^{1/2}, \quad \text{ for all } y \in \mathbb{R}^2.$$

Therefore $u \in W^{1,\infty}(\mathbb{R}^2)$, with $\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq K(1+c)\varepsilon_0^{1/2}$. Differentiating (TW$_c$), we find that $v = \partial_j u$, $j = 1, 2$, satisfies

$$L_\lambda(v) := -\Delta v - \nabla u \nabla v u + c(u \times \partial_1 v) + \lambda v = |\nabla u|^2 v + 2u_3 v_3 u + u_3^2 v - v_3 u - u_3 v + \lambda v, \text{ in } \mathbb{R}^2.$$

Since $\nabla u \in L^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, we deduce that the r.h.s. of the formula above belongs to $L^2(\mathbb{R}^2)$. Therefore taking $\lambda > 0$ large enough, we can invoke the Lax–Milgram theorem to deduce that $v \in H^2(\mathbb{R}^2)$. Then, by the Sobolev embedding theorem, $D^2 u \in L^p(\mathbb{R}^2)$, for all $p \in [2, \infty)$ and a bootstrap argument allows us to conclude that $\nabla u \in W^{k,p}(\mathbb{R}^2)$ for all $k \in \mathbb{N}$ and $p \in [2, \infty]$. In particular, u_3 is uniformly continuous, so that $u_3(x) \to 0$, as $|x| \to \infty$ and $u \in \tilde{\mathcal{E}}(\mathbb{R}^2)$.

The estimates (5.1.13) and (5.1.14) are given by Corollary 5.2.3. □

Now we recall a well-known result that gives the existence of the lifting for any function in $\tilde{\mathcal{E}}(\mathbb{R}^N)$. For the sake of completeness, we sketch its proof.

Lemma 5.2.4. *Let $N \geq 2$ and $v \in \tilde{\mathcal{E}}(\mathbb{R}^N)$. Then there exists $R \geq 0$ such that v admits the lifting*

$$\check{v}(x) = \varrho(x) e^{i\theta(x)}, \text{ on } B(0,R)^c, \tag{5.2.44}$$

where $\varrho = \sqrt{1 - v_3^2}$ and θ is a real-valued function. Moreover, $\varrho, \theta \in H^1_{\mathrm{loc}}(B(0,R)^c)$ and $\nabla \varrho, \nabla \theta \in L^2(B(0,R)^c))$.

Proof. Since $v \in \mathcal{E}(\mathbb{R}^N)$ and $\|v_3\|_{L^\infty(B(0,r)^c)} < 1$, for some $r \geq 0$, the function

$$\psi = \check{v}/|\check{v}|$$

is well-defined on $B(0,r)^c$ and $\psi \in H^1_{\mathrm{loc}}((B(0,r)^c)$. If $r > 0$, invoking the extension theorem for Sobolev spaces, we infer that there exists $\tilde{\psi} \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\tilde{\psi} = \psi$ on $\overline{B}^c_{r+1}$. In the case that $r = 0$, $B(0,r)^c = \mathbb{R}^N$ and ψ is defined globally, but we also denote it by $\tilde{\psi}$ to treat both cases. By results in [13, 17] we have that for every simply connected smooth bounded domain $\Omega \subseteq \mathbb{R}^N$ there exists a real-valued function $\theta_\Omega \in H^1(\Omega)$ such that

$$\tilde{\psi} = e^{i\theta_\Omega}, \qquad \text{on } \Omega.$$

Then, considering $\Omega = B_{r+n} \equiv B(0, r+n)$, $n \geq 1$, we have a sequence of functions $\theta_n \in H^1(B_{r+n})$ such that $\tilde{\psi} = e^{i\theta_n}$ on B_{r+n}. Noticing that

$$\tilde{\psi} = e^{i\theta_n} = e^{i\theta_{n+1}}, \quad \text{on } B_{r+n},$$

we have that $\theta_{n+1}(x) - \theta_n(x) \in 2\pi\mathbb{Z}$, for a.e. $x \in B_{r+n}$. Since a function in $H^1(B_{r+n})$ taking integer values is constant (see [17, Theorem B.1]), we conclude that there exists $k_n \in \mathbb{Z}$ such that $\theta_{n+1} - \theta_n = 2\pi k_n$ on B_{r+n}. Therefore the function

$$\theta(x) = \begin{cases} \theta_1(x), & \text{if } x \in B_{r+1}, \\ \theta_{n+1}(x) - 2\pi \sum_{j=1}^{n} k_j, & \text{if } x \in B_{r+n+1} \setminus B_{r+n},\ n \geq 1, \end{cases}$$

is well-defined and

$$\theta(x) = \theta_{n+1}(x) - 2\pi \sum_{j=1}^{n} k_j, \qquad \text{a.e. on } B_{r+n+1},$$

for all $n \geq 1$. In particular $\theta \in H^1_{\text{loc}}(\mathbb{R}^N)$ and (5.2.44) holds with $R = r$ if $r = 0$ and $R = r + 2$ otherwise. Finally, we notice that

$$|\nabla \check{v}|^2 = \varrho^2 |\nabla\theta|^2 + |\nabla\varrho|^2 \text{ on } B(0,R)^c \tag{5.2.45}$$

and that $1 - \|v_3\|^2_{L^\infty(B(0,R)^c)} = \inf\{\varrho(x)^2 : x \in B(0,R)^c\} > 0$. Since $\nabla v \in L^2(\mathbb{R}^2)$, we conclude that $\nabla\varrho, \nabla\theta \in L^2(B(0,R)^c)$. □

Corollary 5.2.5. *Let $c \geq 0$ and $u \in \mathcal{E}(\mathbb{R}^2)$ be a solution of* (TW_c). *Then there is $R \geq 0$ such that the lifting $\check{u}(x) = \varrho(x)e^{i\theta(x)}$ holds on $B(0,R)^c$ and satisfies $\nabla\varrho, \nabla\theta \in W^{k,p}(B(0,R)^c)$ for any $k \geq 2$ and $p \in [2, \infty]$. Moreover, there exists a constant $\varepsilon(c) > 0$, depending only on c, such that if $E(u) \leq \varepsilon(c)$, then we can take $R = 0$.*

Proof. By Proposition 5.1.5, $u \in \tilde{\mathcal{E}}(\mathbb{R}^2)$ and then Lemma 5.2.4 gives us the existence of the lifting, whose properties follow from Proposition 5.1.5 and (5.2.45). The last assertion is an immediate consequence of (5.2.39). □

In the case $N \geq 3$, some regularity for the solutions of the equation (5.2.1) can be obtained considering that u is a stationary solution in the sense introduced by R. Moser in [80].

Definition 5.2.6. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $g \in L^p(\Omega; \mathbb{R}^3)$. A solution $u \in H^1(\Omega; \mathbb{S}^2)$ of* (5.2.1) *is called* stationary *if*

$$\operatorname{div}(|\nabla u|^2 e_j - 2\nabla u.\partial_j u) = 2\partial_j u \cdot g, \qquad \text{in } \Omega, \tag{5.2.46}$$

for all $j \in \{1, \dots, N\}$ in the distribution sense.

If we suppose that u is a smooth solution of (5.2.1), then

$$\operatorname{div}(|\nabla u|^2 e_j - 2\nabla u.\partial_j u) = -2\Delta u \cdot \partial_j u = 2g \cdot \partial_j u,$$

so it is a stationary solution. However not every solution $u \in H^1(\Omega;\mathbb{S}^2)$ of (5.2.1) satisfies (5.2.46). The advantage of stationary solutions is that they satisfy a monotonicity formula that allows to generalize some standard results for harmonic maps. However, when g belongs only to $L^2(\Omega)$, the regularity estimates hold only for $N \leq 4$.

Lemma 5.2.7 ([80])**.** *Let $N \leq 4$ and $y \in \mathbb{R}^N$. Assume that $u \in H^1(B(y,1);\mathbb{S}^2) \cap W^{1,4}(B(y,1))$ is a stationary solution of* (5.2.1)*, with $\Omega = B(y,1)$ and $g \in L^2(B(y,1))$. Then there exist $K > 0$ and $\varepsilon_0 > 0$, depending only on N, such that if*

$$\|\nabla u\|_{L^2(B(y,1))} + \|g\|_{L^2(B(y,1))} = \varepsilon \leq \varepsilon_0,$$

we have

$$\|\nabla u\|_{L^4(B(y,1/4))} \leq K\varepsilon^{\frac{1}{2}}.$$

Applying this result to equation (TW$_c$), we are led to the following estimate.

Lemma 5.2.8. *Let $N \leq 4$. There exist $K > 0$ and $\varepsilon_0 > 0$, depending only on N, such that for any solution $u \in \mathcal{E}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ of* (TW$_c$)*, with $c \in [0,1]$, satisfying*

$$E(u) \leq \varepsilon_0,$$

we have

$$\|\nabla u\|_{L^4(B(x,1))} \leq KE(u)^{1/2}.$$

Now we are in position to complete the regularity result in higher dimensions stated in the introduction.

Proof of Lemma 5.1.6. Since $u \in UC(\mathbb{R}^N)$ and $\|u_3\|_{L^2(\mathbb{R}^N)} < \infty$, we have that $u \in \tilde{\mathcal{E}}(\mathbb{R}^N)$. Recalling again a classical results for elliptic systems with quadratic growth (see [16, 69, 64]), $u \in UC(\mathbb{R}^N)$ yields that $u \in C^\infty(\mathbb{R}^N)$. This is due to the fact that now we are assuming that u is uniformly continuous and then we can choose $r > 0$ small such that the oscillation of u on the ball $B(y,r)$ is small, uniformly in y. Then we can make the quantity $A(u,r)$ defined in (5.2.23) as small as needed and repeat the first part of the proof of Lemma 5.2.2 to conclude that for all $y \in \mathbb{R}^2$

$$\|D^2 u\|_{L^2(B(y,r/2))} + \|\nabla u\|^2_{L^4(B(y,r/2))} \leq K(N)r^{-1}\left(\|\nabla u\|_{L^2(B(y,r))} + \|u_3\|_{L^2(B(y,r))}\right), \quad (5.2.47)$$

for some constant $K(N)$ and $r > 0$ small enough, independent of y. At this stage we note that we cannot follow the rest of the argument of Lemma 5.2.2, since it relies on the two-dimensional Sobolev embeddings. However, it is well-known that using (5.2.47) it is possible to deduce that $\nabla u \in L^p(\mathbb{R}^N)$, for all $p \geq 2$. More precisely, as discussed before, there exists $r \in (0,1]$ such that

$$\operatorname*{osc}_{B(y,2^N r)} u \leq \frac{1}{8(1+c)(2N-1)}, \text{ for all } y \in \mathbb{R}^N. \quad (5.2.48)$$

Then, by iterating Lemma B.5, we have

$$\int_{B(y,r)} |\nabla u|^{2N+2} \leq K(N)(1+c)^{2N}\frac{E(u)}{r^{2N}}, \quad \text{for all } y \in \mathbb{R}^N. \quad (5.2.49)$$

By proceeding as in the proof of Lemma 5.2.2, we decompose u_i as $u_i = \phi_i + \varphi_i$, where

$$\begin{cases} -\Delta\phi_i = 0, & \text{in } B(y,r), \\ \phi_i = u_i, & \text{on } \partial B(y,r), \end{cases}$$

$$\begin{cases} -\Delta\psi_i = |\nabla u|^2 u_i + u_3^2 u_i - \delta_{i,3}u_3 + cu \times \partial_1 u, & \text{in } B(y,r), \\ \psi_i = 0, & \text{on } \partial B(y,r). \end{cases}$$

In view of (5.2.49), elliptic regularity estimates imply that $\psi_i \in W^{2,N+1}(B(y,r))$ and then by the Sobolev embedding theorem we can establish an upper bound for $\|\nabla\psi_i\|_{L^\infty(B(y,r))}$ in terms of powers of $E(u)$. Since ϕ_i is a harmonic function, we obtain a similar estimate for ϕ_i as in the proof of Lemma 5.2.25. Then we conclude that $\nabla u \in L^\infty(\mathbb{R}^N)$, so that, by interpolation, $\nabla u \in L^p(\mathbb{R}^N)$, for all $p \in [2,\infty]$. Proceeding as in the proof of Proposition 5.1.5, we conclude that $\nabla u \in W^{k,p}(\mathbb{R}^N)$, for all $k \in \mathbb{N}$ and $p \in [2,\infty]$.

Now we turn to (5.1.16) and (5.1.17). Let us first take $N = 3$ and ε_0 given by Lemma 5.2.8, such that $E(u) \le \varepsilon_0$. Then, by the Morrey inequality,

$$\operatorname*{osc}_{B(y,1/2)} u \le K\|\nabla u\|_{L^4(B(y,1))} \le KE(u)^{1/2}, \tag{5.2.50}$$

for all $y \in \mathbb{R}^3$ and for all $c \in [0,1]$. Taking ε_0 smaller if necessary, (5.2.48) holds with $r = 1/16$ and then so it does (5.2.49) (with $r = 1/16$). Hence the previous computations give a bound for $\nabla u \in L^\infty(\mathbb{R}^3)$ depending only on $E(u)$, which yields (5.1.17).

In order to prove (5.1.16), we estimate the minimum of $|u_3|$ on $B(y,1/2)$ as in (5.2.41), and using (5.2.50) we conclude that

$$\max_{B(y,1/2)} |u_3| \le KE(u)^{1/2},$$

which implies (5.1.16).

It only remains to consider the case $N = 4$. Note that the r.h.s. of (5.2.48) is less than or equal to $1/112$, for $c \in [0,1]$. Let $r_* > 0$ be the maximal radius given by the uniform continuity of u for this value, i.e.

$$r_* = \sup\left\{\rho > 0 : \forall x, z \in \mathbb{R}^4, |x - z| \le \rho \Rightarrow |u(x) - u(z)| \le 1/112\right\}.$$

We claim that $r_* \ge 1/2$ for ε_0 small. Arguing by contradiction, we suppose that $0 < r_* < 1/2$. Since (B.2) is satisfied for any $y \in \mathbb{R}^4$, with $r = r_*$ and $s = 2$, Lemma B.5 implies that

$$\|\nabla u\|^6_{L^6(B(y,r_*/2))} \le 8\left(1 + \frac{16}{r_*^2}\right)\|\nabla u\|^4_{L^4(B(y,r_*))}.$$

Since $0 < r_* < 1/2$, the Morrey inequality implies that

$$\operatorname*{osc}_{B(y,r_*/4)} u \le K_1 r_*^{\frac{1}{3}}\|\nabla u\|_{L^6(B(y,r_*/2))} \le K_2 r_*^{\frac{1}{3}}\left(1 + \frac{16}{r_*^2}\right)^{\frac{1}{6}}\|\nabla u\|^{\frac{2}{3}}_{L^4(B(y,r_*))} \le K_3 E(u)^{\frac{1}{3}}, \tag{5.2.51}$$

where we have used Lemma 5.2.8 for the last inequality and $K_3 > 0$ is a universal constant. Finally we notice that there exists a universal constant $\ell \in \mathbb{N}$ such that for any $x \in \mathbb{R}^4$, there is a collection of points $y_1, y_2, \ldots, y_\ell \in \mathbb{R}^4$ such that

$$\operatorname*{osc}_{B(x,2r_*)} u \le \sum_{k=1}^{\ell} \operatorname*{osc}_{B(y_k,r_*/4)} u.$$

Thus, using (5.2.51),

$$\operatorname*{osc}_{B(x,2r_*)} u \leq \ell K_3 E(u)^{\frac{1}{3}}.$$

Taking $\varepsilon_0 \leq 1/(112\ell K_3)^3$, we get that $\operatorname{osc}_{B(x,2r_*)} \leq 1/112$, which contradicts the definition of r_*. Therefore,

$$\operatorname*{osc}_{B(x,1/2)} u \leq 1/112, \quad \text{ for all } x \in \mathbb{R}^4.$$

Moreover, the same argument shows that

$$\operatorname*{osc}_{B(y,1/8)} u \leq KE(u)^{1/3}, \quad \text{ for all } y \in \mathbb{R}^4,$$

and then (5.1.16) and (5.1.17) follow as before. □

5.3 The momentum

5.3.1 Definition of momentum in $\tilde{\mathcal{E}}(\mathbb{R}^N)$

As discussed in the introduction, one of the main technical difficulties in the study of the Landau–Lifshitz equation is to provide a proper definition of momentum in the energy space. In fact, for $N \geq 2$, we formally define the vectorial momentum in the j-direction by

$$P_j(v) = -\frac{1}{N-1}\sum_{k=1}^{N}\int_{\mathbb{R}^N} x_k v\cdot(\partial_j v \times \partial_k v) = -\frac{1}{N-1}\sum_{\substack{k=1\\k\neq j}}^{N}\int_{\mathbb{R}^N} x_k v\cdot(\partial_j v \times \partial_k v),$$

but it is not clear that this quantity is well-defined in $\mathcal{E}(\mathbb{R}^N)$. As proved in Section 5.2, the solutions of (TW_c) that we are considering belong to $\tilde{\mathcal{E}}(\mathbb{R}^N)$ and therefore by Lemma 5.2.4 they admit the lifting

$$v = (\varrho e^{i\theta}, v_3)$$

at infinity. Then, assuming that v is of class C^2, we obtain

$$v\cdot(\partial_j v\times\partial_k v) = v_3\varrho(\partial_j\varrho\partial_k\theta - \partial_k\varrho\partial_j\theta) + \varrho^2(\partial_j\theta\partial_k v_3 - \partial_k\theta\partial_j v_3) = \partial_k(v_3\partial_j\theta) - \partial_j(v_3\partial_k\theta),$$

where we have used that $v_3^2 = 1-\varrho^2$ for the last equality. In consequence, if the lifting for v holds in the whole space, a formal integration by parts yields

$$P_j(v) = \int_{\mathbb{R}^N} v_3\partial_j\theta,$$

which is well-defined in $\mathcal{E}(\mathbb{R}^N)$. However it is not clear how to give a sense to an integration by parts of the type

$$\int_{\mathbb{R}^N} x_k(\partial_k f_1 - \partial_j f_2) = -\int_{\mathbb{R}^N} f_1,$$

$j \neq k$, for a function $f \in L^1(\mathbb{R}^N)$. This difficulty, that also appears in the context of the Gross–Pitaevskii equation, can be handle by defining a *generalized momentum* as an analogue to the definition given in [31] (see also [77]). For the sake of simplicity, we focus only on the first component of the vectorial momentum $p \equiv P_1$ and introduce the notation

$$w_k(v) \equiv v\cdot(\partial_1 v\times\partial_k v), \quad k \in \{2,\dots,N\}.$$

In particular, $w = w_2$.

Definition 5.3.1. *Let $k \in \{2,\ldots,N\}$. We set $\mathbb{L}_k(\mathbb{R}^N) = \{h \in L^1(\mathbb{R}^N) : x_k h \in L^1(\mathbb{R}^N)\}$ and $\mathcal{X}_j(\mathbb{R}^N) = \{\partial_j v \in S'(\mathbb{R}^N) : v \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)\}$, for $j \in \{1,\ldots,N\}$. For any $h_1 \in \mathcal{X}_1(\mathbb{R}^N)$, $h_2 \in \mathcal{X}_k(\mathbb{R}^N)$ and $h_3 \in \mathbb{L}_k(\mathbb{R}^N)$, we define the application L_k on $\mathcal{X}_1(\mathbb{R}^N) + \mathcal{X}_k(\mathbb{R}^N) + \mathbb{L}_k(\mathbb{R}^N)$, by*

$$L_k(h_1 + h_2 + h_3) = \int_{\mathbb{R}^N} x_k h_3 - \int_{\mathbb{R}^N} v,$$

where $\partial_k v = h_2$.

Endowing $\mathbb{L}_k(\mathbb{R}^N)$ and $\mathcal{X}_j(\mathbb{R}^N)$ with the norms

$$\|h\|_{\mathbb{L}_k} = \|h\|_{L^1(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |x_k h|, \quad \|\partial_j v\|_{\mathcal{X}_j} = \|v\|_{L^1(\mathbb{R}^N)} + \|v\|_{L^2(\mathbb{R}^N)},$$

they are Banach spaces. The next result shows that L_k is actually a well-defined linear continuous operator.

Lemma 5.3.2. *Let $k \in \{2,\ldots,N\}$.*

(*a*) *For any $h \in \mathbb{L}_k(\mathbb{R}^N) \cap \mathcal{X}_1(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} x_k h = 0. \tag{5.3.1}$$

(*b*) *For any $h \in \mathbb{L}_k(\mathbb{R}^N) \cap \mathcal{X}_k(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} x_k h = -\int_{\mathbb{R}^N} v,$$

where $\partial_k v = h$.

(*c*) *For any $h \in \mathcal{X}_1(\mathbb{R}^N) \cap \mathcal{X}_k(\mathbb{R}^N)$, such that $h = \partial_1 v = \partial_k u$, for some $v, u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, we have*

$$\int_{\mathbb{R}^N} v = \int_{\mathbb{R}^N} u = 0. \tag{5.3.2}$$

In particular L_k is a well-defined linear continuous operator on $\mathcal{X}_1(\mathbb{R}^N) + \mathcal{X}_k(\mathbb{R}^N) + \mathbb{L}_k(\mathbb{R}^N)$.

Proof. For the proof of (*a*), we consider $v \in L^1(\mathbb{R}^N)$ with $\partial_1 v = h$ and a cut-off function $\eta \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\eta(x) = 1$ if $|x| \leq 1$, $\eta(x) = 0$ if $|x| \geq 2$, and $|\eta'| \leq 2$ on $\mathbb{R}$. Setting $\eta_m(x) = \eta(|x|/m)$ and integrating by parts we have

$$\int_{\mathbb{R}^2} \eta_m(x) x_k h = \int_{\mathbb{R}^2} \partial_1 \eta_m(x) x_k v.$$

Then, since $x_k h \in L^1(\mathbb{R}^N)$, $|\partial_1 \eta_m(x) x_k| \leq 4$ and $\partial_1 \eta_m(x) \to 0$ as $m \to \infty$, invoking the dominated convergence theorem we can pass to the limit $m \to \infty$ and (5.3.1) follows. The proof of (*b*) is similar and thus we omit it.

Now we turn to the proof of (*c*). First we remark that if $v, u \in W^{1,1}(\mathbb{R}^N)$ the result follows as before, using that

$$\int_{\mathbb{R}^N} \eta_m x_1 \partial_1 v = \int_{\mathbb{R}^N} \eta_m x_1 \partial_k u, \qquad \int_{\mathbb{R}^N} \eta_m x_k \partial_1 v = \int_{\mathbb{R}^N} \eta_m x_k \partial_k u,$$

integrating by parts and letting $m \to \infty$. For the general case, we proceed as in [77, Lemma 2.3]. Since $v, u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, we have that the corresponding Fourier transforms $\hat{v}$ and

$\hat{u}$ belong to $C(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, so that (5.3.2) is equivalent to $\hat{v}(0) = \hat{u}(0) = 0$. Arguing by contradiction, let us suppose that $\hat{u}(0) \neq 0$ (the case $\hat{v}(0) \neq 0$ is analogous), then there exist $r_0 > 0$ and $m > 0$ such $|\hat{u}| \geq m$, a.e. on $B(0, r_0)$. By hypothesis $\partial_1 v = \partial_k u$ in $S'(\mathbb{R}^N)$, so that $\xi_1 \hat{v} = \xi_k \hat{u}$ in $L^2(\mathbb{R}^N)$. Therefore,

$$|\hat{v}| \geq \frac{|\xi_k|}{|\xi_1|} m, \qquad \text{a.e. on } B(0, r_0),$$

which contradicts that $\hat{v} \in L^2(\mathbb{R}^N)$. □

In Lemma 5.3.4 below we will prove that

$$w_k(v) \in \mathcal{X}_1(\mathbb{R}^N) + \mathcal{X}_k(\mathbb{R}^N) + \mathbb{L}_k(\mathbb{R}^N), \qquad \text{for all } k \in \{2, \dots, N\}, \tag{5.3.3}$$

for any $v \in \tilde{\mathcal{E}}(\mathbb{R}^N)$. Then we can finally give a proper definition for the momentum.

Definition 5.3.3. *Let $v \in \tilde{\mathcal{E}}(\mathbb{R}^N)$. We define the* generalized momentum *of v as*

$$p(v) = -\frac{1}{N-1} \sum_{k=2}^{N} L_k(w_k(v)).$$

Lemma 5.3.4. *Let $v \in \tilde{\mathcal{E}}(\mathbb{R}^N)$. Then (5.3.3) holds. Moreover, if v satisfies $\|v_3\|_{L^\infty(\mathbb{R}^N)} < 1$, we have*

$$p(v) = \int_{\mathbb{R}^N} v_3 \partial_1 \theta, \tag{5.3.4}$$

where $\check{v} = \sqrt{1 - v_3^2} e^{i\theta}$.

Let $\eta \in C_0^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$ on $\mathbb{R}$, $\eta = 1$ on $B(0,1)$, $\eta = 0$ on $B(0,2)^c$ and $|\eta'| \leq 2$. Setting $\eta_m : \mathbb{R}^N \to \mathbb{R}$ by $\eta_m(x) = \eta(|x|/m)$, $m \in \mathbb{N} \setminus \{0\}$, we have

$$L_k(w_k(v)) = \lim_{m \to \infty} \int_{\mathbb{R}^N} x_k \eta_m w_k(v). \tag{5.3.5}$$

Furthermore, if $v \in \tilde{\mathcal{E}}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$, there exists a sequence $r_n \to \infty$ such that

$$L_k(w_k(v)) = \lim_{r_n \to \infty} \int_{B(0, r_n)} x_k w_k(v), \tag{5.3.6}$$

for all $k \in \{2, \dots, N\}$. In particular,

$$p(v) = -\lim_{r_n \to \infty} \frac{1}{N-1} \sum_{k=2}^{N} \int_{B(0, r_n)} x_k w_k(u).$$

Proof. Let $R \geq 0$ given by Lemma 5.2.4 such that $\check{v} = \varrho e^{i\theta}$ on $B(0,R)^c$. For any $r > 0$ we define two functions $\chi_r, \tilde{\chi}_r \in C^\infty(\mathbb{R}^N)$ such that

$$\chi_r(x) = \begin{cases} 0, & \text{if } |x| \leq r, \\ 1, & \text{if } |x| \geq r+1, \end{cases} \qquad \text{and} \qquad \tilde{\chi}_r = \begin{cases} 0, & \text{if } |x| \leq r+2, \\ 1, & \text{if } |x| \geq r+3, \end{cases} \tag{5.3.7}$$

and for $r > R$ we set

$$\varrho_r = \chi_r \varrho,\ \theta_r = \tilde{\chi}_r \theta,\ \check{v}_r = \varrho_r e^{i\theta_r}, \quad \text{a.e. on } \mathbb{R}^N,$$

so that we extend v to $\mathbb{R}^N$ as $v_r = (\check{v}_r, v_3)$. Clearly, we cannot assure that $|v_r| = 1$ on $\mathbb{R}^N$, but the different supports of χ_r and $\tilde{\chi}_r$ will allow us to overcome this difficulty.

Writing $v = v - v_r + v_r$, we have

$$w_k(v) = \mathcal{V}(v, v_r) + w_k(v_r), \tag{5.3.8}$$

with

$$\mathcal{V}(v, v_r) = v \cdot (\partial_1 v \times \partial_k (v - v_r) + \partial_1 (v - v_r) \times \partial_k v_r) + (v - v_r) \cdot (\partial_1 v_r \times \partial_k v_r).$$

Since $v - v_r$ has compact support, $\mathcal{V}(v, v_r) \in \mathbb{L}_k(\mathbb{R}^N)$. To treat the other term, we compute

$$w_k(v_r) = v_3 \varrho_r (\partial_j \varrho_r \partial_k \theta_r - \partial_k \varrho_r \partial_j \theta_r) + \varrho_r^2 (\partial_j \theta_r \partial_k v_3 - \partial_k \theta_r \partial_j v_3).$$

Although the equality $\varrho_r^2 = 1 - v_3^2$ does not hold on the whole space, this equality is valid on the open set $\{|x| > r + 1\}$ and since $\theta_r \equiv 0$ on $B(0, r + 2)$, we have

$$\varrho_r \partial_l \varrho_r \partial_m \theta_r = -v_3 \partial_l v_3 \partial_m \theta_r, \quad \text{a.e. on } \mathbb{R}^N, \text{ for all } l, m \in \{1, \ldots, N\}.$$

Therefore

$$w_k(v_r) = \partial_k v_3 \partial_1 \theta_r - \partial_1 v_3 \partial_k \theta_r \ \text{ a.e. on } \mathbb{R}^N. \tag{5.3.9}$$

Now we claim that

$$\partial_k v_3 \partial_1 \theta_r - \partial_1 v_3 \partial_k \theta_r = \partial_k (v_3 \partial_1 \theta_r) - \partial_1 (v_3 \partial_k \theta_r) \ \text{ in } S'(\mathbb{R}^N). \tag{5.3.10}$$

To prove (5.3.10), we take a sequence $\theta_{n,r} \in C^2(\mathbb{R}^N)$ such that $\nabla\theta_{n,r} \to \nabla\theta_r$ in $L^2(\mathbb{R}^N)$, as $n \to \infty$. It is clear that (5.3.10) holds with $\theta_{n,r}$ instead of θ_r. On the other hand, since $v_3 \in L^\infty(\mathbb{R}^N)$ and $\nabla v_3 \in L^2(\mathbb{R}^N)$, for any $\phi \in S(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (\partial_k v_3 \partial_1 \theta_{n,r} - \partial_1 v_3 \partial_k \theta_{n,r})\phi \to \int_{\mathbb{R}^N} (\partial_k v_3 \partial_1 \theta_r - \partial_1 v_3 \partial_k \theta_r)\phi, \quad \text{as } n \to \infty,$$

and

$$\begin{aligned}\langle \partial_k (v_3 \partial_1 \theta_{n,r}) - \partial_1 (v_3 \partial_k \theta_{n,r}), \phi \rangle_{S'(\mathbb{R}^N), S(\mathbb{R}^N)} &= -\int_{\mathbb{R}^N} v_3 \partial_1 \theta_{n,r} \partial_k \phi \, dx + \int_{\mathbb{R}^N} v_3 \partial_k \theta_{n,r} \partial_1 \phi \, dx \\ &\to -\int_{\mathbb{R}^N} v_3 \partial_1 \theta_r \partial_k \phi \, dx + \int_{\mathbb{R}^N} v_3 \partial_k \theta_r \partial_1 \phi \, dx,\end{aligned}$$

as $n \to \infty$. Being the l.h.s. of the last two expressions equal, we are led to (5.3.10). Combining with (5.3.8) and (5.3.9), we conclude that

$$w_k(v) = \mathcal{V}(v, v_r) + \partial_k (v_3 \partial_1 \theta_r) - \partial_1 (v_3 \partial_k \theta_r) \quad \text{in } S'(\mathbb{R}^N). \tag{5.3.11}$$

In particular (5.3.3) is satisfied and by Definition 5.3.1 we are led to

$$L_k(w_k(v)) = -\int_{\mathbb{R}^N} v_3 \partial_1 \theta_r + \int_{\mathbb{R}^N} x_k \mathcal{V}(v, v_r). \tag{5.3.12}$$

Since $x_k\eta_m \in S(\mathbb{R}^N)$, from (5.3.11), we deduce that

$$\int_{\mathbb{R}^N} x_k\eta_m w_k(v) = \int_{\mathbb{R}^N} x_k\eta_m \mathcal{V}(v, v_r) - \int_{\mathbb{R}^N} \partial_k(x_k\eta_m)v_3\partial_1\theta_r + \int_{\mathbb{R}^N} \partial_1(x_k\eta_m)v_3\partial_k\theta_r.$$

Since $x_k\mathcal{V}(v, v_r), v_3\partial_k\theta_r \in L^1(\mathbb{R}^N)$, $\partial_j(x_k\eta_m) = \delta_{j,k}\eta_m + x_k\partial_j\eta_m$ and $|x_k\partial_j\eta_m| \leq 4$, the dominated convergence theorem yields that

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} x_k\eta_m w(v) = \int_{\mathbb{R}^N} x_k\mathcal{V}(v, v_r) - \int_{\mathbb{R}^N} v_3\partial_1\theta_r.$$

Comparing with (5.3.12), we conclude that (5.3.5) holds.

In the case that $v \in C^2(\mathbb{R}^N)$, we have that $\theta_r \in C^2(\mathbb{R}^N)$ and (5.3.11) holds in $\mathbb{R}^N$, so that we can multiply (5.3.11) by x_k and integrate by parts on $B(0,\rho)$ to obtain

$$\int_{B(0,\rho)} x_k w_k(v) = -\int_{B(0,\rho)} v_3\partial_1\theta_r + \int_{B(0,\rho)} x_k v_3\mathcal{V}(v, v_r) - \int_{\partial B(0,\rho)} x_k v_3(\partial_k\theta_r\nu_1 - \partial_1\theta_r\nu_k). \tag{5.3.13}$$

Since $v_3\nabla\theta_r \in L^1(\mathbb{R}^N)$, by Lemma B.4 there is a sequence $\rho_n \to \infty$ such that

$$\int_{\partial B(0,\rho_n)} x_k v_3(\partial_k\theta_r\nu_1 - \partial_1\theta_r\nu_k) \to 0.$$

Finally, passing to the limit $\rho_n \to \infty$ in (5.3.13) we obtain

$$\lim_{r_n\to\infty} \int_{B(0,\rho_n)} x_k w_k(v) = -\int_{\mathbb{R}^N} v_3\partial_1\theta_r + \int_{\mathbb{R}^N} \mathcal{V}(v, v_r).$$

Comparing with (5.3.12), (5.3.6) follows.

In the case that $\|v_3\|_{L^\infty(\mathbb{R}^N)} < 1$, $\check{v} = \sqrt{1 - v_3^2}e^{i\theta}$ a.e. on $\mathbb{R}^N$, so that the functions χ_r and $\tilde{\chi}_r$ are not necessary, and we can deduce that (5.3.11) holds with $\mathcal{V} \equiv 0$. Hence (5.3.4) follows immediately. □

5.3.2 Further properties in dimension two

A useful fact of Sobolev spaces is the density of smooth functions with compact support. However, when restricting to functions taking values in $\mathbb{S}^2$ this property is not always preserved. For instance, if $N \geq 3$ we do not have such a result. To see this it is enough to consider $B(0,1) \subset \mathbb{R}^3$ and $v(x) = x/|x| \in H^1(B(0,1))$ that cannot be approximated by smooth functions in the H^1-norm (see [92] for details). On the other hand, in the two-dimensional case, the density result is true (see [92, 20]). More precisely, for $N = 2$ and $A > 0$ we endow $\mathcal{E}(\mathbb{R}^2)$ with the distance

$$d^A_{\mathcal{E}}(u, v) \equiv \left(\|u - v\|^2_{L^2(B(0,A))} + \|\nabla u - \nabla v\|^2_{L^2(\mathbb{R}^2)} + \|u_3 - v_3\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}}. \tag{5.3.14}$$

Note that for $B > A > 0$, it follows from the Sobolev embedding theorem that there exists a constant $K(A, B)$ such that

$$d^A_{\mathcal{E}}(u, v) \leq d^B_{\mathcal{E}}(u, v) \leq K(A, B)d^A_{\mathcal{E}}(u, v), \tag{5.3.15}$$

for any $u, v \in \mathcal{E}(\mathbb{R}^2)$. Therefore the distances $d^A_{\mathcal{E}}$ and $d^B_{\mathcal{E}}$ are equivalent. Now we can state the following density result.

Lemma 5.3.5. *Let $A > 0$ and $v \in \mathcal{E}(\mathbb{R}^2)$. There exists a sequence of functions $(v_n)_{n\in\mathbb{N}}$ in*

$$\mathcal{E}_0^\infty(\mathbb{R}^2) \equiv \mathcal{E}(\mathbb{R}^2) \cap \{f : \mathbb{R}^2 \to \mathbb{R}^3 : \ \exists \lambda \in \mathbb{S}^1 \times \{0\} \text{ s.t. } f - \lambda \in C_0^\infty(\mathbb{R}^2)\},$$

such that

$$d_{\mathcal{E}}^A(v, v_n) \to 0, \text{ as } n \to \infty. \tag{5.3.16}$$

Moreover, if $\|v\|_{L^\infty(B(0,R)^c)} < 1$ for some $R > 0$, then

$$\|v_n\|_{L^\infty(B(0,R+1)^c)} \le \|v\|_{L^\infty(B(0,R)^c)}. \tag{5.3.17}$$

The proof of Lemma 5.3.5 follows the same ideas of the classical result of Schoen and Ulhenbeck [92] (see also [20]), so we omit it.

Corollary 5.3.6. *Let $v \in \tilde{\mathcal{E}}(\mathbb{R}^2)$. Then there exists a sequence of functions $(v_n)_{n\in\mathbb{N}} \in \mathcal{E}_0^\infty(\mathbb{R}^2)$ such that*

$$d_{\mathcal{E}}^A(v, v_n) \to 0 \quad \textit{and} \quad p(v_n) \to p(v), \text{ as } n \to \infty.$$

Proof. Since $v \in \tilde{\mathcal{E}}(\mathbb{R}^2)$, we have that $\delta \equiv \|v_3\|_{L^\infty(B(0,R)^c)} < 1$, for some $R > 0$. In view of Lemma 5.3.5, we only need to verify that the sequence v_n given by Lemma 5.3.5, up to a subsequence, satisfies

$$p(v_n) \to p(v), \text{ as } n \to \infty. \tag{5.3.18}$$

Let $\varepsilon > 0$. By Lemma 5.3.4 there exists an integer $m_\varepsilon > 0$, such that

$$\left| p(v) + \int_{B(0,2m)} x_2 w(v) \eta_m \right| \le \varepsilon/3, \qquad \text{for all } m \ge m_\varepsilon, \tag{5.3.19}$$

and we can assume that $2m_\varepsilon > R + 1$.

To compute the momentum of the sequence v_n, since v_n is smooth and $(v_n)_3 \in C_0^\infty(\mathbb{R}^2)$, using (5.3.6), we get

$$p(v_n) = -\int_{\mathbb{R}^2} x_2 w(v_n) = -\int_{B(0,2m)} x_2 w(v_n)\eta_m - \int_{B(0,m)^c} x_2 w(v_n)(1 - \eta_m), \qquad \text{for all } m \in \mathbb{N}. \tag{5.3.20}$$

Since $d_{\mathcal{E}}^A(v, v_n) \to 0$, there exist functions $f \in L^2(\mathbb{R}^2)$, $g \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ such that, up to a subsequence,

$$|(v_n)_3| \le f, \quad |\nabla (v_n)_j| \le g_j, \text{ for } j \in \{1, 2, 3\}, \text{ for all } n \in \mathbb{N}, \tag{5.3.21}$$

and

$$(v_n)_3 \to v_3, \ \nabla v_n \to \nabla v \ \text{ in } L^2(\mathbb{R}^2), \text{ as } n \to \infty. \tag{5.3.22}$$

Invoking (a modified version of) Lemma B.4, there exists $\tilde{m}_\varepsilon \ge m_\varepsilon$ such that

$$\frac{1}{2\sqrt{1 - \delta^2}} \left(5 \int_{B^c_{\tilde{m}_\varepsilon}} (f^2 + |g|^2) + \rho_\varepsilon \int_{\partial B_{\tilde{m}_\varepsilon}} (f^2 + |g|^2) \right) \le \frac{\varepsilon}{3}, \tag{5.3.23}$$

where $B_{\tilde{m}_\varepsilon} = B(0, \tilde{m}_\varepsilon)$. Since, for $r > R + 1$ we have the lifting $\check{v}_n = \varrho_n e^{i\theta_n}$ on $B(0, R + 1)^c$, then

$$\begin{aligned} \int_{B_r^c} x_2(1 - \eta_m) w(v_n) = &- \int_{B_r^c} (1 - \eta_m - x_2 \partial_2 \eta_m)(v_n)_3 \partial_1 \theta_n - \int_{B_r^c} x_2 \partial_2 \eta_m (v_n)_3 \partial_2 \theta_n \\ &+ \int_{\partial B(0,r)} x_2 (1 - \eta_m)((v_n)_3 \partial_1 \theta_n \nu_2 - (v_n)_3 \partial_1 \theta_n \nu_1). \end{aligned} \tag{5.3.24}$$

Since $|1-\eta_m| \leq 1$, $|\eta_m| \leq 1$, $|x_k \partial_j \eta_m| \leq 4$, by combining with (5.1.9), (5.3.17), (5.3.21), (5.3.22) and (5.3.24), we conclude that

$$\left| \int_{B^c_{\tilde{m}_\varepsilon}} x_2(1-\eta_m)w(v_n) \right| \leq \frac{\varepsilon}{3}, \text{ for all } n, m \in \mathbb{N}. \tag{5.3.25}$$

From (5.3.15), $d^{\tilde{m}_\varepsilon}_{\mathcal{E}}(v, v_n) \to 0$, as $n \to \infty$, so that, up to a subsequence,

$$v_n \to v, \quad \text{a.e. on } B(0, \tilde{m}_\varepsilon). \tag{5.3.26}$$

Hence, (5.3.23) and the dominated convergence theorem imply that there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\left| \int_{B_{\tilde{m}_\varepsilon}} x_2 \eta_m w(v_n) - \int_{B_{\tilde{m}_\varepsilon}} x_2 \eta_m w(v) \right| \leq \frac{\varepsilon}{3}, \quad \text{for all } n \geq n_\varepsilon,\ m \in \mathbb{N}. \tag{5.3.27}$$

Finally, using (5.3.19), (5.3.20), (5.3.25) and (5.3.27) with $m = \tilde{m}_\varepsilon$, we obtain

$$|p(v_n) - p(v)| \leq \varepsilon, \quad \text{for all } n \geq n_\varepsilon,$$

which concludes the proof. ☐

We end this section by showing that the definition of generalized momentum has the expected directional derivative.

Lemma 5.3.7. *Let $v \in \tilde{\mathcal{E}}(\mathbb{R}^2)$ and $\phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^3)$. Then*

$$\lim_{s \to 0} \frac{p(v_s) - p(v)}{s} = \int_{\mathbb{R}^2} \langle \phi_v^T, \partial_1 v \times v \rangle = \int_{\mathbb{R}^2} \langle \phi, \partial_1 v \times v \rangle, \tag{5.3.28}$$

where $v_s = (v + s\phi)/|v + s\phi|$ and $\phi_v^T = \phi - \langle \phi, v \rangle v$ (the tangential component of ϕ).

Proof. First we notice that v_s is well-defined for $|s| < 1/\|\phi\|_{L^\infty(\mathbb{R}^2)}$. Moreover,

$$v_s = v + s\phi_v^T + s^2 g(s, v, \phi),$$

for s small, where g satisfies

$$|g(s, v, \phi)| \leq K(\|\phi\|_{L^\infty(\mathbb{R}^2)})|\phi|,$$

for some constant $K(\|\phi\|_{L^\infty(\mathbb{R}^2)})$. Then

$$w(v_s) = w(v) + s\langle \phi_v^T, \partial_1 v \times \partial_2 v \rangle + \langle v, \partial_1 \phi_v^T \times \partial_2 v \rangle + \langle v, \partial_1 v \times \partial_2 \phi_v^T \rangle + s^2 \tilde{g}(s, v, \phi), \tag{5.3.29}$$

where $\tilde{g}$ satisfies

$$|\tilde{g}(s, v, \phi)| \leq K(\|\phi\|_{W^{1,\infty}(\mathbb{R}^2)})(|\phi||\nabla v| + |\phi||\nabla v|^2 + |\nabla \phi|). \tag{5.3.30}$$

Using some standard identities for the cross product, we check that

$$\langle \phi_v^T, \partial_1 v \times \partial_2 v \rangle = 0, \text{ a.e. on } \mathbb{R}^2.$$

We also notice that

$$\langle v_n, \partial_1\phi_v^T \times \partial_2 v_n\rangle + \langle v_n, \partial_1 v_n \times \partial_2\phi_v^T\rangle = \partial_2\langle \phi_v^T, v_n \times \partial_1 v_n\rangle - \partial_1\langle \phi_v^T, v_n \times \partial_2 v_n\rangle,$$

for any $v_n \in C^2(\mathbb{R}^2)$. In particular, we consider a sequence $v_n \in C^2(\mathbb{R}^2)$ such that $\nabla v_n \to \nabla v$ in $L^2(\mathbb{R}^2)$. Moreover, we can assume that there exists $V \in L^2(\mathbb{R}^2)$ such that

$$|\nabla v_n| \leq V \text{ and } \nabla v_n \to \nabla v, \quad \text{a.e. on } \mathbb{R}^2,$$

so that, by the dominated convergence theorem, we can argue as in the proof of Lemma 5.3.4 to conclude that

$$\langle v, \partial_1\phi_v^T \times \partial_2 v\rangle + \langle v_n, \partial_1 v \times \partial_2\phi_v^T\rangle = \partial_2\langle \phi_v^T, v \times \partial_1 v\rangle - \partial_1\langle \phi_v^T, v \times \partial_2 v\rangle, \text{ in } S'(\mathbb{R}^2).$$

Therefore

$$\frac{w(v_s) - w(v)}{s} = \partial_2\langle \phi_v^T, v \times \partial_1 v\rangle - \partial_1\langle \phi_v^T, v \times \partial_2 v\rangle + s\tilde{g}(s, v, \phi).$$

Noticing that $w(v_s) - w(v)$ and $\tilde{g}$ have compact support, Definition 5.3.3 yields

$$\frac{p(v_s) - p(v)}{s} = -L_2\left(\frac{w(v_s) - w(v)}{s}\right) = \int_{\mathbb{R}^2} \langle \phi_v^T, v \times \partial_1 v\rangle - s\int_{\mathbb{R}^2} x_2\tilde{g}(s, v, \phi). \tag{5.3.31}$$

By (5.3.30), we can invoke again the dominated convergence theorem and pass to the limit $s \to 0$, so that the first equality in (5.3.28) follows. The second one is immediate, since $v \cdot (\partial_1 v \times v) = 0$. ☐

5.4 Pohozaev identities

The generalized momentum defined in Section 5.3 allows us to establish the following Pohozaev identities for (TW_c).

Proposition 5.4.1. *Let $u \in \tilde{\mathcal{E}}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ be a solution of (TW_c). Then we have*

$$E(u) = \int_{\mathbb{R}^2} |\partial_1 u|^2 \, dx, \tag{5.4.1}$$

$$E(u) = \int_{\mathbb{R}^2} |\partial_k u|^2 \, dx - cL_k(w_k(u)), \quad \textit{for all } k \in \{2, \dots, N\}. \tag{5.4.2}$$

Proof. Taking inner product between (TW_c) and $x_k\partial_k u$, $1 \leq k \leq N$, integrating by parts in the ball $B(0, R)$ and using that $u \cdot \partial_k u = 0$, we obtain

$$\int_{B(0,R)} |\partial_k u|^2 - \frac{1}{2}\int_{B(0,R)} |\nabla u|^2 - \int_{\partial B(0,R)} \frac{\partial u}{\partial \nu} \cdot \partial_k u x_k + \int_{\partial B(0,R)} |\nabla u|^2 x_k \nu_k =$$
$$\frac{1}{2}\int_{B(0,R)} u_3^2 - \frac{1}{2}\int_{\partial B(0,R)} u_3^2 x_k \nu_k + c\int_{B(0,R)} (\partial_1 u \times \partial_k u) \cdot u x_k,$$

where ν denotes the exterior normal of the ball $B(0, R)$ and $\frac{\partial u}{\partial \nu} = (\nabla u_1 \cdot \nu, \nabla u_2 \cdot \nu, \nabla u_3 \cdot \nu)$. Then, using Lemma 5.3.4, there is a sequence $r_n \to \infty$ such that

$$E(u) = \int_{\mathbb{R}^N} |\partial_k u|^2 - c\lim_{r_n \to \infty} \int_{\mathbb{R}^N} x_k u \cdot (\partial_1 u \times \partial_k u) = \int_{\mathbb{R}^N} |\partial_k u|^2 - cL_k(w_k(u)),$$

for all $j \in \{1, \dots, N\}$, which completes the proof. ☐

Corollary 5.4.2. *Let $u \in \mathcal{E}(\mathbb{R}^2)$ be a solution of* (TW$_c$). *Then*

$$\int_{\mathbb{R}^2} u_3^2 \, dx = cp(u). \tag{5.4.3}$$

Proof. Writing

$$\int_{\mathbb{R}^2} u_3^2 \, dx = 2E(u) - \int_{\mathbb{R}^2} |\partial_1 u|^2 \, dx - \int_{\mathbb{R}^2} |\partial_2 u|^2 \, dx,$$

since $u \in C^2(\mathbb{R}^2)$ by Proposition 5.1.5, the result is a direct consequence of Proposition 5.4.1. □

5.5 Convolution equations and further integrability

Through this section, we fix $u \in \tilde{\mathcal{E}}(\mathbb{R}^N) \cap UC(\mathbb{R}^N)$ a solution of (TW$_c$) for a speed $c \in [0,1]$. Using the results of Section 5.2, we have that $u \in C^\infty(\mathbb{R}^N)$ and it admits the lifting

$$\check{u} = \varrho e^{i\theta}, \quad \text{a.e. on } B(0,R)^c,$$

for some $R \geq 0$. If $\|u_3\|_{L^\infty(\mathbb{R}^2)} < 1$, we take $R = 0$. Now we recall some definitions given in Subsection 5.1.4. Let $\chi \in C^\infty(\mathbb{R}^N)$ be such that $|\chi| \leq 1$, $\chi = 0$ on $B(0,2R)$ and $\chi = 1$ on $B(0,3R)^c$, if $R > 0$. In the case that $R = 0$, we let $\chi = 1$ on $\mathbb{R}^N$. In this way, we can assume that the function $\chi\theta$ is well-defined on $\mathbb{R}^N$, that

$$G = (G_1, G_2) := u_1 \nabla u_2 - u_2 \nabla u_1 - \nabla(\chi\theta), \quad \text{on } \mathbb{R}^N, \tag{5.5.1}$$

is smooth and that

$$G = -u_3^2 \nabla\theta \quad \text{on } B(0,3R)^c. \tag{5.5.2}$$

Therefore $G \in W^{k,p}(\mathbb{R}^N)$, for all $k \in \mathbb{N}$ and $1 \leq p \leq \infty$.

For $u = (u_1, u_2, u_3)$, equation (TW$_c$) reads

$$-\Delta u_1 = 2e(u)u_1 + c(u_2 \partial_1 u_3 - u_3 \partial_1 u_2), \tag{5.5.3}$$
$$-\Delta u_2 = 2e(u)u_2 + c(u_3 \partial_1 u_1 - u_1 \partial_1 u_3), \tag{5.5.4}$$
$$-\Delta u_3 = 2e(u)u_3 - u_3 + c(u_1 \partial_1 u_2 - u_2 \partial_1 u_1). \tag{5.5.5}$$

Then, using (5.5.3) and (5.5.4),

$$\begin{aligned} \operatorname{div}(G) &= u_1 \Delta u_2 - u_2 \Delta u_1 - \Delta(\chi\theta) \\ &= c(\partial_1 u_3 - u_3 u \cdot \partial_1 u) - \Delta(\chi\theta) \\ &= c\partial_1 u_3 - \Delta(\chi\theta), \end{aligned} \tag{5.5.6}$$

where we used the fact that $u \cdot \partial_1 u = 0$. By combining with (5.5.5), we obtain (5.1.20). Taking Fourier transform in (5.1.20), we get

$$\left(|\xi|^4 + |\xi|^2 - c^2 \xi_1^2\right) \widehat{u}_3(\xi) = |\xi|^2 \widehat{F}(\xi) - c \sum_{j=1}^N \xi_1 \xi_j \widehat{G}_j(\xi), \tag{5.5.7}$$

and hence

$$\widehat{u}_3(\xi) = L_c(\xi) \left(\widehat{F}(\xi) - c\frac{\xi_1^2}{|\xi|^2} \widehat{G}_1(\xi) - c\frac{\xi_1 \xi_2}{|\xi|^2} \widehat{G}_2(\xi) \right), \tag{5.5.8}$$

where

$$L_c(\xi) = \frac{|\xi|^2}{|\xi|^4 + |\xi|^2 - c^2\xi_1^2}.$$

Equivalently, we can write (5.5.8) as the convolution equation (5.1.21) where $\widehat{\mathcal{L}_c} = L_c$. Similarly, from (5.5.6) and (5.5.8), for $j \in \{1, \ldots, N\}$,

$$\partial_j(\chi\theta) = c\,\mathcal{L}_{c,j} * F - c^2 \sum_{k=1}^{N} \mathcal{T}_{c,j,k} * G_k - \sum_{k=1}^{N} \mathcal{R}_{j,k} * G_k, \tag{5.5.9}$$

where

$$\widehat{\mathcal{T}}_{c,j,k} = \frac{\xi_1\xi_j\xi_k}{|\xi|^2(|\xi|^4 + |\xi|^2 - c^2\xi_1^2)} \quad \text{and} \quad \widehat{\mathcal{R}}_{j,k} = \frac{\xi_j\xi_k}{|\xi|^2},$$

for all $j, k \in \{1, \ldots, N\}$. These kernels also appear in the Gross–Pitaevskii equation and their properties are studied in [48, 51, 30]. In order to obtain some estimates, a key element is the following multiplier theorem due to Lizorkin.

Theorem 5.5.1 ([75]). *Let $m \in C^N(\mathbb{R}^N \setminus \{0\})$ and $\alpha \in [0, 1)$. Suppose that exists $M > 0$ such that*

$$\sup \left\{ |\xi_1^{k_1+\alpha} \cdots \xi_N^{k_N+\alpha} D^k m(\xi)| \ : \ \xi \in \mathbb{R}^N \setminus \{0\}, \ k \in \{0, 1\}^N \right\} \leq M.$$

Then m is a Fourier multiplier from $L^p(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$, for all $q \in (1/(1-\alpha), \infty)$, where $1/p = \alpha + 1/q$. More precisely, there exists a positive constant $K(N, \alpha, q)$, such that

$$\|\mathcal{M} * f\|_{L^q(\mathbb{R}^N)} \leq K(N, \alpha, q) M \|f\|_{L^p(\mathbb{R}^N)}, \ \forall f \in L^p(\mathbb{R}^N), \tag{5.5.10}$$

where $m = \widehat{\mathcal{M}}$.

Proposition 5.5.2. *Let $\alpha \in [0, 1)$ and $c \in [0, 1)$. Then L_c is a Fourier multiplier from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$, with $1/p = 1/q + \alpha$. Moreover, there exists a positive constant $K(\alpha, q)$ independent of c such that*

$$\|\mathcal{L}_c * f\|_{L^q(\mathbb{R}^2)} \leq K(\alpha, q) \|f\|_{L^p(\mathbb{R}^2)}, \ \forall f \in L^p(\mathbb{R}^2), \quad \text{if } \alpha \in [2/3, 1), \tag{5.5.11}$$

$$\|\mathcal{L}_c * f\|_{L^q(\mathbb{R}^2)} \leq \frac{K(\alpha, q)}{(1-c^2)^3} \|f\|_{L^p(\mathbb{R}^2)}, \ \forall f \in L^p(\mathbb{R}^2), \quad \text{if } \alpha \in [0, 2/3). \tag{5.5.12}$$

Proof. In the case that $\alpha \in [2/3, 1)$, the result is a particular case of [30, Lemma 3.3]. Moreover, from [30] we also have

$$|\xi_1|^{k_1+\alpha}|\xi_2|^{k_2+\alpha}|D^k L_c(\xi)| \leq \frac{K}{|\xi|^{2-2\alpha}}, \quad \text{for all } |\xi| \geq 1, \ k \in \{0, 1\}^2. \tag{5.5.13}$$

We also note that

$$|\xi|^4 + |\xi|^2 - c^2\xi_1^2 = |\xi|^4 + (1-c^2)\xi_1^2 + \xi_2^2 \geq (1-c^2)|\xi|^2. \tag{5.5.14}$$

Thus

$$|L_c(\xi)| \leq \frac{1}{1-c^2}. \tag{5.5.15}$$

Now we compute for $i = 1, 2$

$$\partial_i L_c(\xi) = \frac{2\xi_i}{(|\xi|^4 + |\xi|^2 - c^2\xi_1^2)^2}\Big(- |\xi|^4 - c^2\xi_1^2 + c^2\delta_{1,i}|\xi|^2\Big). \tag{5.5.16}$$

Therefore, for $0 < |\xi| \leq 1$, using that $|\xi|^4 \leq |\xi|^2$ and (5.5.14),

$$|\xi_i|^{1+\alpha}|\partial_i L_c(\xi)| \leq \frac{2|\xi|^{4+\alpha}(1 + 2c^2)}{|\xi|^4(1 - c^2)^2} \leq \frac{K}{(1 - c^2)^2}. \tag{5.5.17}$$

A similar computation yields

$$|\xi_1|^{1+\alpha}|\xi_2|^{1+\alpha}|\partial^2_{12} L_c(\xi)| \leq \frac{K}{(1 - c^2)^3}, \quad \text{for all } 0 < |\xi| \leq 1. \tag{5.5.18}$$

By putting together (5.5.13), (5.5.15)–(5.5.18) and invoking Theorem 5.5.1, we obtain (5.5.12). □

In the case of the L^2-norm, the following computation will be useful.

Lemma 5.5.3. *For any $c \in [0, 1)$, we have*

$$\|L_c\|_{L^2(\mathbb{R}^2)} = \frac{\pi}{(1 - c^2)^{1/4}}.$$

Proof. Using polar coordinates, we obtain

$$\|L_c\|^2_{L^2(\mathbb{R}^2)} = \int_0^\infty \int_0^{2\pi} \frac{r\, d\theta\, dr}{(r^2 + 1 - c^2\cos^2(\theta))^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{1 - c^2\cos^2(\theta)}.$$

Then, using the change of variables $t = \tan(\theta)$, straightforward computations yield the result. □

As a consequence, we obtain an improvement on the integrability of u_3 that until now belongs only to $L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

Proposition 5.5.4. *Let $c \in [0, 1)$. For any $p \in (1, 2)$, we have that $u_3 \in L^p(\mathbb{R}^2)$. Moreover, if $\|u_3\|_{L^\infty(\mathbb{R}^2)} < 1$, then*

$$\|u_3\|_{L^q(\mathbb{R}^2)} \leq K(\alpha, q)\|u_3\|_{L^\infty(\mathbb{R}^2)} E(u)^{\frac{1}{2}+\frac{\alpha}{2}+\frac{1}{2q}}, \quad \text{if } \alpha \in [2/3, 1), \tag{5.5.19}$$

$$\|u_3\|_{L^q(\mathbb{R}^2)} \leq \frac{K(\alpha, q)}{(1 - c^2)^3}\|u_3\|_{L^\infty(\mathbb{R}^2)} E(u)^{\frac{1}{2}+\frac{\alpha}{2}+\frac{1}{2q}}, \quad \text{if } \alpha \in [0, 2/3), \tag{5.5.20}$$

for all $q \in (1/(1 - \alpha), \infty)$. Furthermore,

$$\|u_3\|_{L^2(\mathbb{R}^2)} \leq \frac{K}{(1 - c^2)^{1/4}}\|u_3\|_{L^\infty(\mathbb{R}^2)} E(u). \tag{5.5.21}$$

Proof. Let us recall that by Proposition 5.1.5, $F, G_1, G_2 \in L^p(\mathbb{R}^2)$, for all $p \in [1, \infty]$. On the other hand, from the Riesz-operator theory we have that the functions $\xi \mapsto \xi_i\xi_j/|\xi|^2$ are L^q-multipliers for any $q \in (1, \infty)$ and $1 \leq i, j \leq 2$. Thus by (5.5.8) and invoking Proposition 5.5.2 with $\alpha = 0$, we conclude that $u_3 \in L^q(\mathbb{R}^2)$, for all $q \in (1, \infty)$.

Under the assumption $\|u_3\|_{L^\infty(\mathbb{R}^2)} < 1$, we have $G = -u_3^2 \nabla\theta$ on $\mathbb{R}^2$. Then, by Corollary 5.2.3 and (5.1.9),

$$\begin{aligned}\|F\|_{L^r(\mathbb{R}^2)} + \|G_j\|_{L^r(\mathbb{R}^2)} &\leq K\|u_3\|_{L^\infty(\mathbb{R}^2)}\|e(u)\|_{L^r(\mathbb{R}^2)}\\ &\leq K\|u_3\|_{L^\infty(\mathbb{R}^2)}\|e(u)\|^{1-1/r}_{L^\infty(\mathbb{R}^2)}\|e(u)\|^{1/r}_{L^1(\mathbb{R}^2)} \leq K\|u_3\|_{L^\infty(\mathbb{R}^2)}E(u)^{1/2(1+1/r)},\end{aligned}$$

for any $r \in [1,\infty)$ and $1 \leq j \leq 2$. Therefore, combining with (5.1.21), (5.5.11) and (5.5.12), we obtain (5.5.19) and (5.5.20).

Finally, applying the Plancherel identity to (5.5.8) and using Lemma 5.5.3

$$\begin{aligned}\|u_3\|_{L^2(\mathbb{R}^2)} &\leq \|L_c\|_{L^2(\mathbb{R}^2)}\left(\|\widehat{F}\|_{L^\infty(\mathbb{R}^2)} + \|\widehat{G}_1\|_{L^\infty(\mathbb{R}^2)} + \|\widehat{G}_2\|_{L^\infty(\mathbb{R}^2)}\right)\\ &\leq \|L_c\|_{L^2(\mathbb{R}^2)}\left(\|F\|_{L^1(\mathbb{R}^2)} + \|G_1\|_{L^1(\mathbb{R}^2)} + \|G_2\|_{L^1(\mathbb{R}^2)}\right)\\ &\leq \frac{K}{(1-c^2)^{1/4}}\|u_3\|_{L^\infty(\mathbb{R}^2)}E(u),\end{aligned}$$

which established inequality (5.5.21). □

In the case $N \geq 3$, a similar analysis can be made, including the critical value $c = 1$.

Proposition 5.5.5. *Let $N \geq 3$ and $c \in (0,1]$. Then $u_3 \in L^p(\mathbb{R}^N)$, for all $p \in (1,2)$. Moreover, if $\|u_3\|_{L^\infty(\mathbb{R}^N)} < 1$, we have*

$$\|u_3\|_{L^2(\mathbb{R}^N)} \leq K(N)\|u_3\|_{L^\infty(\mathbb{R}^2)}\left(1 + \|\nabla u_3\|^{\frac{2N-5}{2(2N-1)}}_{L^\infty(\mathbb{R}^N)}\right)E(u)^{\frac{2N+3}{2(2N-1)}}. \tag{5.5.22}$$

Proof. Let us recall that from [30, Lemma 4.3],

$$\|\mathcal{L}_c * f\|_{L^2(\mathbb{R}^N)} \leq K(N)\|f\|_{L^{\frac{2(2N-1)}{2N+3}}(\mathbb{R}^N)}.$$

Then, the same computations as those in the proof of Proposition 5.5.4, yield

$$\begin{aligned}\|u_3\|_{L^2(\mathbb{R}^N)} &\leq K(N)\left(\|F\|_{L^{\frac{2(2N-1)}{2N+3}}(\mathbb{R}^N)} + \sum_{j=1}^N \|G_j\|_{L^{\frac{2(2N-1)}{2N+3}}(\mathbb{R}^N)}\right)\\ &\leq K(N)\|u_3 e(u)\|_{L^{\frac{2(2N-1)}{2N+3}}(\mathbb{R}^N)}\\ &\leq K(N)\|u_3\|_{L^\infty(\mathbb{R}^N)}\|e(u)\|^{\frac{2N-5}{2(2N-1)}}_{L^\infty(\mathbb{R}^N)}\|e(u)\|^{\frac{2N+3}{2(2N-1)}}_{L^1(\mathbb{R}^N)},\end{aligned}$$

and since $|u_3| \leq 1$, we are led to (5.5.22). Finally, the same type of computations as in the proof of Proposition 5.5.2 allows us to apply Theorem 5.5.1 to deduce that L_c is an L^p-multiplier, for all $q \in (1,\infty)$. Then we conclude that $u_3 \in L^p(\mathbb{R}^N)$, for all $p \in (1,2)$ in a similar way to the proof of Proposition 5.5.4. □

Lemma 5.5.6. *For all $k \in \mathbb{N}$ and $p \in (1,\infty]$, we have $u_3, \nabla(\chi\theta) \in W^{k,p}(\mathbb{R}^N)$.*

Proof. By Proposition 5.1.5 and Lemma 5.1.6, it remains only to treat the case $p \in (1,2)$. Differentiating (5.1.21) and (5.5.9), we have

$$\partial^\alpha u_3 = \mathcal{L}_c * \partial^\alpha F - c\sum_{j=1}^N \mathcal{L}_{c,j} * \partial^\alpha G_j,$$
$$\partial^\alpha \partial_j(\chi\theta) = c\,\mathcal{L}_{c,j} * \partial^\alpha F - c^2\sum_{k=1}^N \mathcal{T}_{c,j,k} * \partial^\alpha G_k - \sum_{k=1}^N \mathcal{R}_{j,k} * \partial^\alpha G_k,$$

for all $\alpha \in \mathbb{N}^N$. The conclusion follows by observing that $\mathcal{L}_{c,j}, \mathcal{T}_{c,j,k}$ and $\mathcal{R}_{j,k}$ are L^p-multipliers for all $p \in (1,\infty)$, that $u_3, \nabla(\chi\theta), \nabla u \in W^{k,p}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$ and $p \in [2,\infty)$ and using the Leibniz rule. □

Corollary 5.5.7. *Let $N \geq 2$ and $c \in [0,1)$. Then the function θ is bounded on $B(0,R)^c$ and there exists $\bar\theta \in \mathbb{R}$ such that*

$$\theta(x) \to \bar\theta, \quad \text{as } |x| \to \infty. \tag{5.5.23}$$

Proof. By Lemma 5.5.6, $\nabla\theta \in L^p(\mathbb{R}^N)$, for all $1 < p \leq \infty$. Then there exists $\bar\theta \in \mathbb{R}$ such that $\theta - \bar\theta \in L^{\frac{Np}{N-p}}(\mathbb{R}^N)$ (see e.g. [59, Theorem 4.5.9]). Since $\nabla\theta \in L^\infty(\mathbb{R}^N)$, $\theta \in UC(\mathbb{R}^N)$ and therefore (5.5.23) follows. □

Proof of Proposition 5.1.8. For $c = 0$, we deduce from (5.4.1) and (5.4.2) that $\|u_3\|_{L^2(\mathbb{R}^N)} = 0$, so that $u_3 \equiv 0$. Thus $\check{u} = e^{i\theta}$ on $\mathbb{R}^N$ and using (TW$_c$) (see (5.6.2) below) we deduce that $\Delta\theta = 0$ on $\mathbb{R}^N$. Therefore, by Corollary 5.5.7, we have that θ is a bounded harmonic function, which implies that it is constant and so that $\check{u}$ is a constant function taking values in $\mathbb{S}^1$. □

5.6 Properties of vortexless solutions

In this section we assume that $c \in (0,1]$ and that the corresponding nontrivial solution $u \in \tilde{\mathcal{E}}(\mathbb{R}^N) \cap UC(\mathbb{R}^N)$ of (TW$_c$) satisfies

$$\|u_3\|_{L^\infty(\mathbb{R}^N)} < 1. \tag{5.6.1}$$

This implies that $\check{u} = \varrho e^{i\theta}$, on $\mathbb{R}^2$, and therefore we can recast (TW$_c$) as

$$\operatorname{div}(\varrho^2\nabla\theta) = c\partial_1 u_3, \tag{5.6.2}$$
$$-\Delta\varrho + \varrho|\nabla\theta|^2 = 2e(u)\varrho - cu_3\varrho\partial_1\theta, \tag{5.6.3}$$
$$-\Delta u_3 = (2e(u)-1)u_3 + c\varrho^2\partial_1\theta. \tag{5.6.4}$$

From these equations we obtain the following useful integral relations.

Lemma 5.6.1. *Assume that* (5.6.1) *holds. Then we have the following identities*

$$\int_{\mathbb{R}^N} \varrho^2|\nabla\theta|^2 = c\int_{\mathbb{R}^N} u_3\partial_1\theta, \tag{5.6.5}$$

$$\int_{\mathbb{R}^N}|\nabla\varrho|^2 + \int_{\mathbb{R}^N}\varrho^2|\nabla\theta|^2 = 2\int_{\mathbb{R}^N} e(u)\varrho^2 - c\int_{\mathbb{R}^N} u_3\varrho^2\partial_1\theta, \tag{5.6.6}$$

$$2\int_{\mathbb{R}^N}\varrho|\nabla\varrho|^2 + 2\int_{\mathbb{R}^N} e(u){u_3}^2\varrho = \int_{\mathbb{R}^N}\varrho u_3^2|\nabla\theta|^2 + c\int_{\mathbb{R}^N}\varrho u_3^3\partial_1\theta, \tag{5.6.7}$$

$$\int_{\mathbb{R}^N}|\nabla u_3|^2 + \int_{\mathbb{R}^N} u_3^2 = 2\int_{\mathbb{R}^N} e(u)u_3^2 + c\int_{\mathbb{R}^N}\varrho^2 u_3\partial_1\theta. \tag{5.6.8}$$

Proof. First we recall that by Lemma B.4, for any $f\in L^2(\mathbb{R}^2)$, there exists a sequence $R_n\to\infty$ such that

$$\int_{\partial B(0,R_n)}|f| \le \frac{(2\pi)^{1/2}}{(\ln(R_n))^{1/2}}. \tag{5.6.9}$$

Now we multiply (5.6.2) by θ and integrate by parts on the ball $B(0,R_n)$. Using the fact that $u_3,\nabla\theta\in L^2(\mathbb{R}^N)$ and $u_3,\varrho,\theta\in L^\infty(\mathbb{R}^N)$, we can choose R_n as in (5.6.9) such that the integrals on $\partial B(0,R_n)$ go to zero and (5.6.5) follows.

To obtain (5.6.6), (5.6.7) and (5.6.8), we multiply (5.6.3) by ϱ, (5.6.3) by u_3^2 and (5.6.4) by u_3, and proceed in a similar way. ☐

Now we give the proof of a more precise version of Proposition 5.1.11.

Proposition 5.6.2. *Assume that* $\|u_3\|_{L^\infty(\mathbb{R}^2)}\le 1/2$. *Then*

$$\|u_3\|_{L^\infty(\mathbb{R}^2)} \ge \frac{\sqrt{2}}{\sqrt{7}}\sqrt{1-c^2} \tag{5.6.10}$$

and

$$E(u) \le \frac{1}{2}\left(\frac{31}{3}\|u_3\|^2_{L^\infty(\mathbb{R}^2)} + 1 + c\right)\int_{\mathbb{R}^N} u_3^2. \tag{5.6.11}$$

In particular, for all $L>1$, *there exists* $\varepsilon(L)>0$ *such that if* $E(u)\le\bar\varepsilon(L)$, *then*

$$E(u)\le Lp(u). \tag{5.6.12}$$

Proof. Let $\delta = \|u_3\|_{L^\infty(\mathbb{R}^2)}\in[0,1/2]$. First we note that Corollary 5.4.2 and (5.6.5) imply

$$\int_{\mathbb{R}^2}\varrho^2|\nabla\theta|^2 = c\int_{\mathbb{R}^2} u_3\partial_1\theta = \int_{\mathbb{R}^2} u_3^2. \tag{5.6.13}$$

From (5.6.7) and the Cauchy–Schwarz inequality, we obtain

$$2\sqrt{1-\delta^2}\int_{\mathbb{R}^2}|\nabla\varrho|^2 \le \frac{\delta^2}{\sqrt{1-\delta^2}}\int_{\mathbb{R}^2}(\varrho\nabla\theta)^2 + c\delta^2\left(\int u_3^2\right)^{\frac{1}{2}}\left(\int(\varrho^2\nabla\theta)^2\right)^{\frac{1}{2}}. \tag{5.6.14}$$

Using (5.6.13) and the fact that

$$\frac{1}{\sqrt{1-\delta^2}}\le 2, \tag{5.6.15}$$

we conclude that

$$\int_{\mathbb{R}^N} |\nabla \varrho|^2 \leq \delta^2(2+c)\int_{\mathbb{R}^N} u_3^2 \leq 3\delta^2 \int_{\mathbb{R}^N} u_3^2. \tag{5.6.16}$$

Similarly, from (5.6.8), (5.6.13), (5.6.16) and the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_3|^2 + \int_{\mathbb{R}^N} u_3^2 &\leq \delta^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + u_3^2) + c\left(\int_{\mathbb{R}^N} (\varrho \partial_1 \theta)^2\right)^{1/2} \left(\int_{\mathbb{R}^N} u_3^2\right)^{1/2} \\ &\leq \delta^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + u_3^2) + c\int_{\mathbb{R}^N} u_3^2.\end{aligned} \tag{5.6.17}$$

Combining with (5.6.13) and (5.6.16), we obtain

$$\big(1-\delta^2\big)\int_{\mathbb{R}^N} |\nabla u_3|^2 \leq \big(\delta^2(1+3\delta^2) - (1-c)\big)\int_{\mathbb{R}^N} u_3^2.$$

Since $\delta \leq 1/2$ and $c \in (0,1]$, we can simplify the estimate as

$$\int_{\mathbb{R}^N} |\nabla u_3|^2 \leq \frac{4}{3}\left(\frac{7\delta^2}{4} - (1-c)\right)\int_{\mathbb{R}^N} u_3^2 \leq \frac{7\delta^2}{3}\int_{\mathbb{R}^N} u_3^2. \tag{5.6.18}$$

In particular, we see that

$$\frac{7\delta^2}{4} \geq (1-c)$$

is a necessary condition for the existence of nontrivial solutions. Writing

$$1-c = \frac{1-c^2}{1+c} \geq \frac{1-c^2}{2},$$

(5.6.10) follows. By plugging (5.6.13), (5.6.16) and (5.6.18) in (5.6.17), we have

$$\int_{\mathbb{R}^N} |\nabla u_3|^2 + \int_{\mathbb{R}^N} u_3^2 \leq \left(\frac{22\delta^2}{3} + c\right)\int_{\mathbb{R}^N} u_3^2,$$

which combined with (5.6.13) and (5.6.16), yields

$$E(u) \leq \frac{1}{2}\left(\frac{31\delta^2}{3} + 1 + c\right)\int_{\mathbb{R}^N} u_3^2.$$

Using that $c \leq 1$ and invoking again Corollary 5.4.2 and (5.2.39), there exists $\bar{\varepsilon}$ such that

$$E(u) \leq (K(\bar{\varepsilon})E(u) + 1)p(u),$$

provided that $E(u) \leq \bar{\varepsilon}$, which yields (5.6.12). □

In the higher dimensional case, our estimate is not as precise as in the case $N = 2$, but it is sufficient for our purposes.

Proposition 5.6.3. *Assume that* $\|u_3\|_{L^\infty(\mathbb{R}^N)} \leq 1/2$. *Then*

$$E(u) \leq 3\int_{\mathbb{R}^N} u_3^2. \tag{5.6.19}$$

Proof. The main difference with the proof of Proposition 5.6.2 is that (5.6.13) is no longer valid. However setting $\delta = \|u_3\|_{L^\infty(\mathbb{R}^2)} \in (0, 1/2]$, by the Cauchy–Schwarz inequality we have

$$\int_{\mathbb{R}^N} u_3 \partial_1 \theta \le \left(\int_{\mathbb{R}^N} u_3^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} (\partial_1 \theta)^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{1-\delta^2}} \left(\int_{\mathbb{R}^N} u_3^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} (\varrho^2 \nabla \theta)^2\right)^{\frac{1}{2}}.$$

Thus from (5.6.5),

$$\int_{\mathbb{R}^N} \rho^2 |\nabla \theta|^2 \le \frac{c^2}{1-\delta^2} \int_{\mathbb{R}^N} u_3^2 \le \frac{4}{3} \int_{\mathbb{R}^N} u_3^2. \tag{5.6.20}$$

Combining with (5.6.14) (that is also valid for $N \ge 2$) and (5.6.15), we are led to

$$\int_{\mathbb{R}^N} |\nabla \rho|^2 \le \left(\frac{2}{3} + \frac{1}{2\sqrt{3}}\right) \int_{\mathbb{R}^N} u_3^2. \tag{5.6.21}$$

Similarly, using the first inequality in (5.6.17), (5.6.20) and (5.6.21), we deduce that

$$\int_{\mathbb{R}^N} |\nabla u_3|^2 \le \frac{4}{3} \left(\frac{1}{4} \left(\frac{2}{3} + \frac{1}{2\sqrt{3}} + \frac{4}{3} \right) + \left(\frac{4}{3}\right)^{1/2} - 1 \right) \int_{\mathbb{R}^N} u_3^2 \le \int_{\mathbb{R}^N} u_3^2. \tag{5.6.22}$$

Finally, by putting together (5.6.20), (5.6.21) and (5.6.22),

$$E(u) \le \frac{1}{2} \left(\frac{4}{3} + \frac{2}{3} + \frac{1}{2\sqrt{3}} + 2 \right) \int_{\mathbb{R}^N} u_3^2 \le 3 \int_{\mathbb{R}^N} u_3^2.$$

□

To keep our notation short, now we set

$$\epsilon \equiv \sqrt{1-c^2}.$$

Then the Pohozaev identities and (5.6.5) give the following estimate.

Proposition 5.6.4. *Let $N = 2$ and assume that $\|u_3\|_{L^\infty(\mathbb{R}^N)} \le 1/2$. Suppose also that $E(u) \le \varepsilon_1$, for some $\varepsilon_1 > 0$. Then*

$$p(u) - E(u) + \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\nabla u_3|^2}{1-u_3^2} = p(u) \frac{\epsilon^2}{1+\sqrt{1-\epsilon^2}}, \tag{5.6.23}$$

In particular, if there exists $M \ge 0$ such that $E(u) \le p(u) + M\epsilon^2$, we have

$$\int_{\mathbb{R}^2} |\nabla u_3|^2 \le 2(p(u) + M)\epsilon^2. \tag{5.6.24}$$

Proof. By adding (5.4.3) and (5.6.5), we obtain

$$E(u) - cp(u) = \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\nabla u_3|^2}{1-u_3^2}.$$

Then, using the definition of ϵ we are led to (5.6.23). If $E(u) \le p(u) + M\epsilon^2$, inequality (5.6.24) is a direct consequence of (5.6.23). □

Using estimate (5.6.24), we can improve (5.5.21) as follows.

Proposition 5.6.5. *Let $N = 2$, $c \in (0, 1)$ and $E(u) \leq \varepsilon_1$, $\|u_3\|_{L^\infty(\mathbb{R}^2)} \leq 1/2$. Assume that there exists a constant $M \geq 0$ such that $E(u) \leq p(u) + M\epsilon^2$. Then*

$$\|u_3\|_{L^\infty(\mathbb{R}^2)} \leq K(M, p, \varepsilon_1)\epsilon^{2/p}, \quad \text{for all } p \in (2, \infty), \tag{5.6.25}$$

and

$$\|u_3\|_{L^2(\mathbb{R}^2)} \leq K(M, p, \varepsilon_1)\epsilon^{2/p-1/2}E(u), \quad \text{for all } p \in (2, \infty). \tag{5.6.26}$$

Proof. By the Morrey inequality, for any $p > 2$, there exists a constant $K(p)$ such that

$$|u_3(x) - u_3(y)| \leq K(p)|x - y|^{1-2/p}\|\nabla u_3\|_{L^p(\mathbb{R}^2)}, \tag{5.6.27}$$

for all $x, y \in \mathbb{R}^2$. Since $E(u) \leq \varepsilon_1$, in view of (5.1.9), we have $p(u) \leq 2\varepsilon_1/\sqrt{3}$. Hence, using (5.2.40) and (5.6.24), for any $p \in (2, \infty)$,

$$\|\nabla u_3\|_{L^p(\mathbb{R}^2)} \leq \|\nabla u_3\|_{L^\infty(\mathbb{R}^2)}\|\nabla u_3\|^{2/p}_{L^2(\mathbb{R}^2)} \leq K(M, \varepsilon_1)2^{1+1/p}\varepsilon_1^{1/4+1/p}\epsilon^{2/p}. \tag{5.6.28}$$

By combining (5.6.27) and (5.6.28) we obtain

$$\|u_3\|_{L^\infty(\mathbb{R}^2)} - K2^{1+1/p}\varepsilon_1^{1/4+1/p}\epsilon^{2/p}|x - y|^{1-2/p} \leq |u_3(y)|.$$

At this stage we fix the radius r as

$$r^{1-2/p} = \frac{\|u_3\|_{L^\infty(\mathbb{R}^2)}}{K2^{2+1/p}\varepsilon_1^{1/4+1/p}\epsilon(u)^{2/p}},$$

so that

$$\frac{1}{2}\|u_3\|_{L^\infty(\mathbb{R}^2)} \leq |u_3(y)|, \quad \text{for all } y \in B(x, r). \tag{5.6.29}$$

Then, integrating on the ball $B(x, r)$ we conclude that

$$\begin{aligned}\left(\int_{B(x,r)} |u_3(y)|^q\right)^{\frac{1}{q}} &\geq \frac{\pi^{\frac{1}{q}}}{2}\|u_3\|_{L^\infty(\mathbb{R}^2)}r^{\frac{2}{q}} \\ &\geq K(p, q, \varepsilon_1)\|u_3\|_{L^\infty(\mathbb{R}^2)}\left(\|u_3\|_{L^\infty(\mathbb{R}^2)}\epsilon(u)^{-\frac{2}{p}}\right)^{\frac{2p}{q(p-2)}},\end{aligned} \tag{5.6.30}$$

where we have used (5.6.29). On the other hand, letting $\alpha = 2/3$ and $q = 4$ in Proposition 5.5.4, we have

$$\|u_3\|_{L^4(\mathbb{R}^2)} \leq K(M, \varepsilon_1)\|u_3\|_{L^\infty(\mathbb{R}^2)}. \tag{5.6.31}$$

Taking $q = 4$ in (5.6.30) and combining with (5.6.31), we conclude (5.6.25). Finally, (5.6.26) follows from (5.5.21) and (5.6.25). □

At this point we dispose of all the elements to prove our nonexistence results.

Proof of Theorem 5.1.7. Let ε_0 given by Lemma 5.1.6 such that if $E(u) \leq \varepsilon_0$, then (5.1.16) and (5.1.17) hold. In the case that $E(u) \geq \varepsilon_0$, (5.1.18) is satisfied taking $\mu \leq \varepsilon_0$. By combining Proposition 5.5.5 and estimate (5.6.19) in Proposition 5.6.3, we conclude that

$$E(u) \leq 3\|u_3\|^2_{L^2(\mathbb{R}^N)} \leq K(N)E(u)^{\frac{2N+3}{2N-1}}. \tag{5.6.32}$$

Thus, since u is nonconstant, $E(u) > 0$ and so that

$$K \leq K(N)^{\frac{-(2N-1)}{4}} \leq E(u), \tag{5.6.33}$$

for some $K > 0$. Therefore, taking $\mu = \min\{\varepsilon_0, K\}$, the proof is complete. □

Proof of Theorem 5.1.9. By virtue of (5.2.39), we can fix $\varepsilon_1 > 0$ such that if $E(u) \leq \varepsilon_1$, then $\|u_3\|_{L^\infty(\mathbb{R}^2)} \leq 1/2$. In the case that $E(u) \geq \varepsilon_1$, (5.1.19) is satisfied taking $\kappa_M \leq \varepsilon_1$. Otherwise, $E(u) \leq \varepsilon_1$ and we can invoke Proposition 5.6.5 with $p = 3$, so that

$$\|u_3\|_{L^2(\mathbb{R}^2)} \leq K(M, \varepsilon_1)\epsilon(u)^{1/6}E(u). \tag{5.6.34}$$

Also, since $\|u_3\|^2_{L^\infty(\mathbb{R}^2)} \leq 1/4$ and $c < 1$, (5.6.11) in Proposition 5.6.2 yields

$$E(u) \leq 3\|u_3\|^2_{L^2(\mathbb{R}^2)}. \tag{5.6.35}$$

Using (5.6.34) and (5.6.35), we get

$$E(u) \leq K(M, \varepsilon_1)\epsilon^{1/3}E(u)^2.$$

Since u is nonconstant, $E(u) > 0$ and then

$$E(u) \geq \frac{1}{K(M, \varepsilon_1)(1 - c^2)^{1/6}} \geq \frac{1}{K(M, \varepsilon_1)}.$$

Setting

$$\kappa_M = \min\left\{\varepsilon_1, \frac{1}{K(M, \varepsilon_1)}\right\},$$

the conclusion follows. □

5.7 Decay at infinity

In this section we provide some further analysis on the behavior at infinity for finite energy traveling waves with speed $c \in (0, 1)$. In view of (5.1.21) and (5.5.9), the ideas developed in [15, 29] allow us to study the decay at infinity. Moreover, since the kernels in (5.1.21) and (5.5.9) are the same as those for the traveling waves for the Gross–Pitaevskii equation, we can apply several results obtained by P. Gravejat [48, 50, 51] to the equation (TW$_c$).

The key step is to obtain some decay at infinity of the solutions of (TW$_c$). This can be achieved following the arguments in [11].

Proposition 5.7.1. *Assume that $c \in (0, 1)$. Let $u \in \mathcal{E}(\mathbb{R}^N)$ be a solution of* (TW$_c$)*. Suppose further that $u \in UC(\mathbb{R}^N)$ if $N \geq 3$. Then there exist constants $R_1, \alpha > 0$ such that for all $R \geq R_1$,*

$$\int_{B(0,R)^c} e(u) \leq \left(\frac{R_1}{R}\right)^\alpha \int_{B(0,R_1)^c} e(u). \tag{5.7.1}$$

Proof. By Corollary 5.2.5, there exists $R_0 > 0$ such that equations (5.6.2)–(5.6.4) hold on $B(0, R_0^c)$. Let $\rho > r \geq R_0$ and

$$\Omega_{r,\rho} = \{r \leq |x| \leq \rho\}.$$

Multiplying (5.6.2) by $\theta - \theta_r$, with $\theta_r = \frac{1}{|\partial B_r|}\int_{\partial B_r} \theta$, and integrating by parts, we get

$$\int_{\Omega_{r,\rho}} \varrho^2 \nabla\theta^2 = c\int_{\Omega_{r,\rho}} u_3\partial_1\theta + \int_{\partial\Omega_{r,\rho}} (\theta-\theta_r)\varrho^2\partial_\nu\theta - c\int_{\partial\Omega_{r,\rho}} (\theta-\theta_r)u_3\nu_1, \tag{5.7.2}$$

where ν denotes the outward normal to $\Omega_{r,\rho}$.

We recall that the Poincaré inequality for ∂B_r reads

$$\int_{\partial B_r} (\theta-\theta_r)^2 \leq r^2 \int_{\partial B_r} |\nabla_\tau\theta|^2. \tag{5.7.3}$$

Then we obtain

$$\left|\int_{\partial B_r} (\theta-\theta_r)\varrho^2\partial_\nu\theta\right| \leq r\left(\int_{\partial B_r} |\nabla\theta|^2\right)^{1/2}\left(\int_{\partial B_r} |\rho\nabla\theta|^2\right)^{1/2} \leq \frac{r}{\sqrt{1-\delta^2}}\int_{\partial B_r} |\rho\nabla\theta|^2,$$

where $\delta = \|u_3\|_{L^\infty(B_r^c)}$. Similarly, using also the inequality $ab \leq a^2/2 + b^2/2$,

$$\left|\int_{\partial\Omega_{r,\rho}} (\theta-\theta_r)u_3\nu_1\right| \leq \frac{r}{\sqrt{1-\delta^2}}\int_{\partial B_r} e(u) \quad \text{and} \quad \left|\int_{B_r^c} u_3\partial_1\theta\right| \leq \frac{1}{\sqrt{1-\delta^2}}\int_{B_r^c} e(u).$$

On the other hand, by Lemma 5.5.6 and Corollary 5.5.7,

$$(\theta-\theta_r)\varrho^2\partial_\nu\theta, (\theta-\theta_r)u_3\nu_1 \in L^2(B(0,R_0)^c).$$

Then by Lemma B.4, we conclude that there exists a sequence $\rho_n \to \infty$ such that

$$\int_{\partial B_{\rho_n}} (\theta-\theta_r)\varrho^2\partial_\nu\theta \to 0 \quad \text{and} \quad \int_{\partial B_{\rho_n}} (\theta-\theta_r)u_3\nu_1 \to 0. \tag{5.7.4}$$

Therefore, taking $\rho = \rho_n$, using (5.7.2)–(5.7.4) and the dominated convergence theorem we conclude that

$$\int_{B_r^c} \varrho^2\nabla\theta^2 \leq \frac{c}{\sqrt{1-\delta^2}}\int_{B_r^c} e(u) + \frac{3r}{\sqrt{1-\delta^2}}\int_{\partial B_r} e(u).$$

In the same way, multiplying (5.6.4) by u_3, integrating by parts on the set $\Omega_{r,\tilde\rho_n}$, for a suitable sequence $\tilde\rho_n \to \infty$, we are led to

$$\int_{B_r^c} (|\nabla u_3|^2 + u_3^2) \leq (2\delta^2 + c)\int_{B_r^c} e(u) + \int_{\partial B_r} e(u).$$

Since $c < 1$, we can choose r large enough such that

$$\frac{1}{2(1-\delta^2)}\left(2\delta^2 + c\left(1 + \frac{1}{\sqrt{1-\delta^2}}\right)\right) < 1.$$

Therefore, noticing that

$$e(u) \leq \frac{1}{2(1-\delta^2)}(|\nabla u_3|^2 + \varrho^2|\nabla\theta|^2 + u_3^2), \tag{5.7.5}$$

we conclude that there exists a constant $K(\delta, c) > 0$ such that

$$\int_{B_r^c} e(u) \leq K(\delta, c) r \int_{\partial B_r} e(u). \tag{5.7.6}$$

Since

$$\frac{d}{dr} \int_{B_r^c} e(u) = - \int_{\partial B_r} e(u),$$

we can integrate inequality (5.7.6) to conclude that

$$\int_{B_R^c} e(u) \leq \Big(\frac{r}{R}\Big)^{1/K(c,\delta)} \int_{B_r^c} e(u), \quad \text{for all } R \geq r,$$

which completes the proof. $\square$

Corollary 5.7.2. *Under the hypotheses and notations of Proposition 5.7.1, we have*

$$|\cdot|^\beta e(u) \in L^1(\mathbb{R}^N) \quad \textit{and} \quad |\cdot|^\beta (|F| + |G_1| + |G_2|) \in L^1(\mathbb{R}^N),$$

for all $\beta \in [0, \alpha)$.

Proof. Since $u \in C^\infty(\mathbb{R}^N)$, the fact that $|\cdot|^\beta e(u) \in L^1(\mathbb{R}^2)$ is a direct consequence of Proposition 5.7.1 (see e.g. [48, Proposition 28]).

On the other hand, we take R large enough such that $\|u_3\|_{L^\infty(B_R^c)} \leq 3/4$. Then using that $|u_3| \leq 1$ and (5.5.2) we deduce that

$$|F| + |G_1| + |G_2| \leq 2e(u) + \frac{u_3^2}{2} + \frac{|\nabla \theta|^2}{2} \leq 2e(u) + \frac{u_3^2}{2} + \varrho^2 |\nabla \theta|^2 \leq 5e(u),$$

and then the conclusion follows. $\square$

The properties of the kernels appearing in equations (5.1.21) and (5.5.9) has been extensively studied in [48]. Indeed, using the sets

$$\mathbb{M}_k(\mathbb{R}^N) = \Big\{ f : \mathbb{R}^N \to \mathbb{C} : \sup_{x \in \mathbb{R}^N} |x|^k |f(x)| < \infty \Big\}, \quad k \in \mathbb{N},$$
$$\mathbb{M}(\mathbb{R}^N) = \Big\{ f \in C^\infty(\mathbb{R}^N \setminus \{0\}; \mathbb{C}) : D^k f \in \mathbb{M}_k(\mathbb{R}^N) \cap \mathbb{M}_{k+2}(\mathbb{R}^N), \text{ for all } k \in \mathbb{N} \Big\},$$

it is proved that

$$D^n \mathcal{L}_c, D^n \mathcal{L}_{c,j}, D^n \mathcal{T}_{c,j,k} \in \mathbb{M}_{\alpha+n}(\mathbb{R}^N), \text{ for all } 1 \leq j,k \leq N,\ n \in \mathbb{N},\ \alpha \in (N-2, N], \tag{5.7.7}$$

$$\widehat{\mathcal{L}}_c, \widehat{\mathcal{L}}_{c,j}, \widehat{\mathcal{T}}_{c,j,k} \in \mathbb{M}(\mathbb{R}^N). \tag{5.7.8}$$

Similar results hold for the composed Riesz kernels $\mathcal{R}_{j,k}$. Combining these results with Corollary 5.7.2, equations (5.1.21) and (5.5.9) allow us to obtain the following algebraic decay.

Lemma 5.7.3. *For any* $n \in \mathbb{N}$,

$$u_3, D^n(\nabla(\chi\theta)), D^n(\nabla \check{u}) \in \mathbb{M}_N(\mathbb{R}^N) \quad \textit{and} \quad D^n u_3 \in \mathbb{M}_{N+1}(\mathbb{R}^N).$$

Proof. In view of Corollary 5.7.2, the proof follows using the same arguments in [48, Theorem 11]. □

Proof of Proposition 5.1.12. Inequality (5.1.23) and the estimate for ∇u_3 in (5.1.24) are particular cases of Lemma 5.7.3. A sightly improvement of Lemma 5.7.3 is necessary for the decay of the second derivatives in (5.1.24) and (5.1.25). This can be done by following the lines in [50, Theorem 6], which completes the proof. □

The pointwise convergence at infinity follows from the general arguments in [50], valid for all functions satisfying (5.7.8).

Lemma 5.7.4 ([50]). *Assume that T is a tempered distribution whose Fourier transform $\widehat{T} = P/Q$ is a rational fraction which belongs to $\mathbb{M}(\mathbb{R}^N)$ and such that $Q \neq 0$ on $\mathbb{R}^N \setminus \{0\}$. Then there exists a function $T_\infty \in L^\infty(\mathbb{S}^{N-1};\mathbb{C})$ such that*

$$R^N T(R\sigma) \to T_\infty(\sigma), \quad \text{as } R \to \infty, \quad \text{for all } \sigma \in \mathbb{S}^{N-1}.$$

Moreover, assume that $f \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap \mathbb{M}_{2N}(\mathbb{R}^N)$. Then $g \equiv T * f$ satisfies

$$R^N g(R\sigma) \to T_\infty(\sigma) \int_{\mathbb{R}^N} f(x)\,dx, \quad \text{as } R \to \infty, \quad \text{for all } \sigma \in \mathbb{S}^{N-1}.$$

Roughly speaking, it only remains to pass to the limit in the terms associated to the Riesz kernels $\mathcal{R}_{i,j}$. For this purpose, we also recall the following.

Lemma 5.7.5 ([50]). *Assume that $f \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap \mathbb{M}_{2N}(\mathbb{R}^N)$ with $\nabla f \in L^\infty(\mathbb{R}^N) \cap \mathbb{M}_{2N+1}(\mathbb{R}^N)$. Then $g \equiv \mathcal{R}_{j,k} * f$ satisfies for all $j,k \in \{1,\dots,N\}$,*

$$R^N g(R\sigma) \to (2\pi)^{-\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)(\delta_{j,k} - N\sigma_j\sigma_k) \int_{\mathbb{R}^N} f(x)\,dx, \quad \text{as } R \to \infty, \quad \text{for all } \sigma \in \mathbb{S}^{N-1}.$$

Finally, we provide a sketch of the proof of Theorem 5.1.13.

Proof of Theorem 5.1.13. In view of (5.1.21), (5.7.8) and Lemma 5.7.3, we can apply Lemma 5.7.4 to the function u_3 to conclude that there exists $u_{3,\infty} \in L^\infty(\mathbb{S}^{N-1};\mathbb{R})$ such that

$$R^N u_3(R\sigma) \to u_{3,\infty}(\sigma), \text{ as } R \to \infty, \text{ for all } \sigma \in \mathbb{S}^{N-1}, \tag{5.7.9}$$

where

$$u_{3,\infty}(\sigma) = \mathcal{L}_{c,\infty}(\sigma) \int_{\mathbb{R}^N} F - c \sum_{j=1}^{N} \mathcal{L}_{c,j,\infty}(\sigma) \int_{\mathbb{R}^N} G_j, \tag{5.7.10}$$

for some functions $\mathcal{L}_{c,\infty}, \mathcal{L}_{c,j,\infty}$. Moreover, adapting [51, Proposition 2], we obtain

$$\begin{aligned} \mathcal{L}_{c,\infty}(\sigma) &= \frac{\Gamma\left(\frac{N}{2}\right)(1-c^2)^{\frac{N-3}{2}} c^2}{2\pi^{\frac{N}{2}}(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}} \left(1 - \frac{N\sigma_1^2}{1-c^2+c^2\sigma_1^2}\right), \\ \mathcal{L}_{c,j,\infty}(\sigma) &= \frac{\Gamma\left(\frac{N}{2}\right)(1-c^2)^{\frac{N-1}{2}}}{2\pi^{\frac{N}{2}}(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}} \left(\delta_{j,1}(1-c^2)^{-\frac{\delta_{j,1}+1}{2}} - \frac{N(1-c^2)^{-\delta_{j,1}}\sigma_1\sigma_j}{1-c^2+c^2\sigma_1^2}\right), \end{aligned} \tag{5.7.11}$$

which gives (5.1.29).

Now we turn to equation (5.5.9). Proceeding as before and using also Lemma 5.7.5, we infer that there exist functions $\theta^j_\infty \in L^\infty(\mathbb{S}^{N-1};\mathbb{R})$, $j \in \{1,\dots,N\}$ such that

$$R^N \partial_j \theta(R\sigma) \to \theta^j_\infty(\sigma), \quad \text{ as } R \to \infty,$$

for all $j \in \{1,\dots,N\}$, and also that θ^j_∞ is given by

$$\theta^j_\infty(\sigma) = c\mathcal{L}_{c,j,\infty}(\sigma) \int_{\mathbb{R}^N} F - \sum_{k=1}^N \left(c^2 \mathcal{T}_{c,j,k,\infty}(\sigma) + \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}(\delta_{j,k} - N\sigma_j\sigma_k) \right) \int_{\mathbb{R}^N} G_k. \tag{5.7.12}$$

As before, adapting [51, Proposition 2] we have

$$\begin{aligned}\mathcal{T}_{c,j,k,\infty} = \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}c^2} \Bigg((1-c^2)^{\frac{N}{2}} \Bigg(&\frac{\delta_{j,k}(1-c^2)^{-\frac{\delta_{j,1}+\delta_{k,1}+1}{2}}}{(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}} - \frac{N(1-c^2)^{-\delta_{j,1}-\delta_{k,1}+\frac{1}{2}}\sigma_j\sigma_k}{(1-c^2+c^2\sigma_1^2)^{\frac{N+2}{2}}} \Bigg) \\ &-\delta_{j,k} + N\sigma_j\sigma_k \Bigg).\end{aligned} \tag{5.7.13}$$

At this stage, we invoke Corollary 5.5.7 and suppose that $\bar\theta = 0$. Then by [50, Lemma 10],

$$R\theta(R\sigma) \to \theta_\infty(\sigma) := -\frac{1}{N-1}\sum_{j=1}^N \sigma_j \theta^j_\infty, \quad \text{ as } R \to \infty. \tag{5.7.14}$$

A further analysis shows that the convergence in (5.7.9) and (5.7.14) are uniform, which implies that

$$R^{N-1}(\check u(R\sigma) - 1) = R^{N-1}\left(\sqrt{1-u_3^2(R\sigma)}\exp(i\theta(R\sigma)) - 1 \right) \to i\theta_\infty(\sigma), \quad \text{ in } L^\infty(\mathbb{S}^{N-1}).$$

By combining with the expression for θ^j_∞ above, (5.1.26) follows with $\lambda_\infty = 1$ and $\check u_\infty = \theta_\infty$, provided that $\bar\theta = 0$. Moreover, using (5.7.10)–(5.7.14) and that

$$\begin{aligned}\sum_{j=1}^N \sigma_j \mathcal{L}_{c,j,\infty}(\sigma) &= -\frac{\Gamma\left(\frac{N}{2}\right)(N-1)(1-c^2)^{\frac{N-3}{2}}\sigma_1}{2\pi^{\frac{N}{2}}(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}}, \\ \sum_{j=1}^N \sigma_j \mathcal{T}_{c,j,k,\infty}(\sigma) &= -\frac{\Gamma\left(\frac{N}{2}\right)(N-1)\sigma_k}{2\pi^{\frac{N}{2}}c^2}\left(\frac{(1-c^2)^{\frac{N}{2}-\frac{1}{2}-\delta_{k,1}}}{(1-c^2+c^2\sigma_1^2)^{\frac{N}{2}}} - 1 \right),\end{aligned}$$

we obtain (5.1.28).

In the case that $\bar\theta \neq 0$, it is enough to redefine the function G in (5.5.1) as

$$G = u_1\nabla u_2 - u_2 \nabla u_1 - \nabla(\chi(\theta - \bar\theta))$$

and then we can establish an equation such as (5.5.9) for $\partial_j(\chi(\theta-\bar\theta))$. Since $\theta(x) - \bar\theta \to 0$, as $x \to \infty$, we conclude as before that there exists $\theta_\infty \in L^\infty(\mathbb{S}^{N-1};\mathbb{R})$ such that

$$R^{N-1}\left(\sqrt{1-u_3^2(R\sigma)}\exp(i(\theta(R\sigma) - \bar\theta) - 1 \right) \to i\theta_\infty(\sigma), \quad \text{ in } L^\infty(\mathbb{S}^{N-1}).$$

Since $\sqrt{1-u_3^2(R\sigma)}\exp(i(\theta(R\sigma)-\bar\theta) = \check u(R\sigma)\exp(-i\bar\theta)$, taking $\lambda_\infty = \exp(i\bar\theta)$, we conclude that

$$R^{N-1}(\check u(R\sigma) - \lambda_\infty) \to i\lambda_\infty\theta_\infty, \quad \text{ in } L^\infty(\mathbb{S}^{N-1}),$$

which completes the proof of Theorem 5.1.13. □

5.8 The minimizing curve in dimension two

The purpose of this section is to prove Theorem 5.1.1 and Proposition 5.1.3. We do this by a series of lemmas.

Lemma 5.8.1. *Let $A > 0$ and $\lambda \in \mathbb{S}^1 \times \{0\}$. Given any number $\mathfrak{s} > 0$, there exists a sequence of nonconstant functions $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{E}_0^\infty(\mathbb{R}^2)$, with $v_n - \lambda \in C_0^\infty(\mathbb{R}^2)$, satisfying*

$$d(v_n) = 0,\ p(v_n) = \mathfrak{s},\ d_{\mathcal{E}}^A(v_n, \lambda) \leq K(A)\sqrt{\mathfrak{s}},\ \text{and}\ E(v_n) \to \mathfrak{s},\ \text{as}\ n \to \infty.$$

In particular, $E_{\min}^0(\mathfrak{p}) \leq \mathfrak{p}$, for any $\mathfrak{p} \geq 0$, and the map $\mathfrak{p} \mapsto \Xi(\mathfrak{p})$ is nonnegative.

Proof. Let $\lambda = (\cos(\theta), \sin(\theta), 0)$ for some real number θ. For $\varphi, \psi \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$, we define

$$v \equiv \big(\sqrt{1-\psi^2}\cos(\theta+\varphi), \sqrt{1-\psi^2}\sin(\theta+\varphi), \psi\big).$$

Assuming further that $\|\psi\|_{L^\infty(\mathbb{R}^2)} < 1/2$, we have that $\sqrt{1-\psi^2}$ is well-defined and smooth on $\mathbb{R}^2$. As a consequence, the function v is also a well-defined smooth function and we can compute

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^2}\Big(\frac{|\nabla\psi|^2}{1-\psi^2} + \big(1-\psi^2\big)|\nabla\varphi|^2 + \psi^2\Big) \quad \text{and} \quad p(v) = \int_{\mathbb{R}^2}\psi\partial_1\varphi.$$

We next introduce the rescaled functions

$$v_{\mu,\nu}^n(x_1, x_2) \equiv \big(\sqrt{1-\mu^2\psi^2}\cos(\theta+\nu\varphi), \sqrt{1-\mu^2\psi^2}\sin(\theta+\nu\varphi), \mu\psi\big)\Big(\frac{x_1}{n}, \frac{x_2}{n^2}\Big), \quad \text{for } \mu, \nu \in \mathbb{R}.$$

Then

$$E(v_{\mu,\nu}^n) = \frac{n^2}{2}\int_{\mathbb{R}^2}\Big(\frac{\mu^2(|\partial_1\psi|^2 + n^{-2}|\partial_2\psi|^2)}{n(1-\mu^2\psi^2)} + \frac{\nu^2}{n}\big(1-\mu^2\psi^2\big)\big(|\partial_1\varphi|^2 + n^{-2}|\partial_2\varphi|^2\big) + n\mu^2\psi^2\Big), \quad (5.8.1)$$

and, by Lemma 5.3.4, we can compute the momentum as

$$p(v_{\mu,\nu}^n) = n^2\mu\nu\int_{\mathbb{R}^2}\psi\partial_1\varphi. \quad (5.8.2)$$

Finally, we look for parameters $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$, such that the conclusions of Lemma 5.8.1 hold.

Concerning the energy, we first notice that the terms including derivatives in the x_2-direction are negligible with respect to the terms including derivatives in the x_1-direction. Assuming furthermore that μ and ν are small parameters (so that the momentum remains bounded as $n \to \infty$), we deduce that, at least formally,

$$E(v_{\mu,\nu}^n) \simeq \frac{n^2}{2}\int_{\mathbb{R}^2}\Big(\frac{\nu^2}{n}|\partial_1\varphi|^2 + n\mu^2\psi^2\Big), \quad (5.8.3)$$

as $n \to \infty$. Recalling that we search parameters such that $E(v_{\mu,\nu}^n) \simeq p(v_{\mu,\nu}^n)$ (and that the inequality $2ab \leq a^2 + b^2$ holds with equality if and only if $a = b$), formulas (5.8.2) and (5.8.3) suggest to choose the function $\psi = \partial_1\varphi$ and the parameter $\nu = n\mu$, then to fix the value of μ so that $p(v_{\mu,n\mu}^n) = \mathfrak{s}$, i.e.

$$\mu \equiv \Big(\frac{\mathfrak{s}}{n^3\int_{\mathbb{R}^2}(\partial_1\varphi)^2}\Big)^{\frac{1}{2}}. \quad (5.8.4)$$

Consequently, $\mu \geq 1$ for n is large enough, the map $v_n \equiv v^n_{\mu,n\mu}$ belongs to $\tilde{\mathcal{E}}(\mathbb{R}^2)$, with $v_n - \lambda \in C_c^\infty(\mathbb{R}^2)$ and $p(v_n) = \mathfrak{s}$. Combining (5.8.1) with (5.8.4), its energy satisfies

$$E(v_n) \to \mathfrak{s}, \text{ as } n \to \infty.$$

Since the function φ is chosen such that $\|\psi\|_{L^\infty(\mathbb{R}^2)} = \|\partial_1\varphi\|_{L^\infty(\mathbb{R}^2)} < 1/2$, the function v_n also takes values in the set $\{x \in \mathbb{S}^2 : |x_3| < \mu/2\}$. Therefore, the topological degree of the smooth map $v_n \circ \Pi$, where Π refers to the stereographic projection, is equal to 0 for n large enough. As a consequence, the quantity $d(v_n)$ is equal to 0 for n large.

In order to complete the proof of Lemma 5.8.1, it remains to estimate the distance $d_{\mathcal{E}}^A(v_n, \lambda)$. In this direction, combining (5.8.1) and (5.8.4) with the fact that $\mu\|\partial_1\varphi\|_{L^\infty(\mathbb{R}^2)} < 1/2$ for n large enough, we infer that

$$\|\nabla v_n\|^2_{L^2(\mathbb{R}^2)} + \|(v_n)_3\|^2_{L^2(\mathbb{R}^2)} \leq 2E(v_n) \leq 3n^3\mu^2 \int_{\mathbb{R}^2} \big(|\nabla\partial_1\varphi|^2 + |\nabla\varphi|^2\big) \leq K(\varphi)\mathfrak{s}, \tag{5.8.5}$$

for some $K(\varphi) > 0$. On the other hand, we can write

$$|v_n - \lambda|^2 = 2\big(1 - (1 - \mu^2|\partial_1\varphi|^2)^{\frac{1}{2}} \cos(n\mu\varphi)\big) \leq \mu^2\Big(2|\partial_1\varphi|^2 + n^2\varphi^2\Big),$$

so that

$$\int_{B(0,A)} |v_n - \lambda|^2 \leq \mu^2 n^3 \int_{E_A^n} \big(2|\partial_1\varphi|^2 + n^2\varphi^2\big),$$

where $E_A^n \equiv \{y \in \mathbb{R}^2 : y_1^2 + n^2y_2^2 < A^2/n^2\}$. In view of (5.8.4), it follows that

$$\int_{B(0,A)} |v_n - \lambda|^2 \leq \mu^2 n^3 \bigg(2\int_{\mathbb{R}^2} |\partial_1\varphi|^2 + \frac{\pi A^2}{n}\|\varphi\|^2_{L^\infty(\mathbb{R}^2)}\bigg) \leq K(A,\varphi)\mathfrak{s},$$

for some $K(A,\varphi)$. Given that φ is fixed, we drop the dependence on φ. We deduce from (5.8.5) and the last estimate that there exists some constant $K(A)$, such that

$$d_{\mathcal{E}}^A(v_n, \lambda) \leq K(A)\sqrt{\mathfrak{s}}.$$

This concludes the proof of Lemma 5.8.1, since the last assertions in Lemma 5.8.1 are immediate consequences of the definitions of $E^0_{\min}$ and Ξ. □

Lemma 5.8.2. *Let $v \in \tilde{\mathcal{E}}(\mathbb{R}^2)$ with $d(v) = 0$. There exists a sequence of maps $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{E}_0^\infty(\mathbb{R}^2)$ such that $d(v_n) = 0$, $p(v_n) = p(v)$,*

$$v_n \to v \text{ in } \mathcal{E}(\mathbb{R}^2) \quad \text{and} \quad E(v_n) \to E(v), \text{ as } n \to \infty. \tag{5.8.6}$$

In particular, given any $\mathfrak{p} \geq 0$,

$$E^0_{\min}(\mathfrak{p}) = \inf\big\{E(v) : v \in \tilde{\mathcal{E}}(\mathbb{R}^2),\ v - \lambda \in C_0^\infty(\mathbb{R}^2),\ \lambda \in \mathbb{S}^1 \times \{0\},\ d(v) = 0 \text{ and } p(v) = \mathfrak{p},\ \big\}.$$

Proof. By Lemma 5.3.5 there exist functions $(\mathfrak{v}_n)_{n\in\mathbb{N}}$ in $\mathcal{E}_0^\infty(\mathbb{R}^2)$ such that $\mathfrak{v}_n - \lambda_n \in C_0^\infty(\mathbb{R}^2)$ for some vector $\lambda_n \in \mathbb{S}^1 \times \{0\}$, satisfying

$$d_{\mathcal{E}}^A(v, \mathfrak{v}_n) \to 0, \quad \text{and} \quad p(\mathfrak{v}_n) \to p(v), \quad \text{as } n \to \infty. \tag{5.8.7}$$

In particular $E(\mathfrak{v}_n) \to E(v)$, as $n \to \infty$ and by Lemma B.6,

$$d(\mathfrak{v}_n) \to d(v) = 0, \quad \text{as } n \to \infty. \tag{5.8.8}$$

In the case $p(v) \neq 0$, we introduce the rescaled functions v_n defined by

$$v_n(x_1, x_2) \equiv \mathfrak{v}_n(x_1, \mu_n x_2),$$

where $\mu_n \equiv p(\mathfrak{v}_n)/p(v)$. In this way, in view of (5.8.7), $\mu_n \to 1$, and $v_n \to v$ in $\mathcal{E}(\mathbb{R}^2)$, which implies (5.8.6) and also that $d(v_n) \to d(v)$. Since d is an integer-valued function, we can assume, up to a subsequence, that $d(v_n) = 0$ for any $n \in \mathbb{N}$. In regard of Lemma 5.3.4, a straightforward computation shows that

$$p(v_n) = -\int_{\mathbb{R}^2} x_2 w(v_n) = -\frac{1}{\mu_n}\int_{\mathbb{R}^2} x_2 w(\mathfrak{v}_n) = p(v).$$

This concludes the proof of the first assertion in Lemma 5.8.2 in the case $p(v) \neq 0$.

When $p(v) = 0$, the proof is slightly more involved. In this case, we may assume that $p(\mathfrak{v}_n) \neq 0$ for n sufficiently large. Otherwise, up to a subsequence, the conclusion holds with $v_n = \mathfrak{v}_n$. As before, we can assume that $d(\mathfrak{v}_n) = 0$ for any $n \in \mathbb{N}$. Given two positive numbers $A > 0$ and $\delta > 0$, we invoke Lemma 5.8.1 to construct a map $w_\delta^n \in \mathcal{E}(\mathbb{R}^2)$, with $w_\delta^n - \lambda_n \in C_0^\infty(\mathbb{R}^2)$, such that

$$d(w_\delta^n) = 0,\ p(w_\delta^n) = \delta,\ E(w_\delta^n) \leq |\delta|,\ \text{and}\ d_{\mathcal{E}}^A(w_\delta^n, \lambda_n) \leq K(A)\sqrt{|\delta|},$$

for some constant $K(A) > 0$. Denoting $\tilde{w}_\delta^n(x_1, x_2) = w_\delta^n(-x_1, x_2)$, this construction is also possible for any $\delta < 0$. We then consider the map v_n defined by

$$v_n = \begin{cases} \mathfrak{v}_n, & \text{on } \operatorname{supp}(\mathfrak{v}_n - \lambda_n), \\ w_{\delta_n}^n(\cdot - a_n), & \text{on } \operatorname{supp}(w_{\delta_n}^n(\cdot - a_n) - \lambda_n), \\ \lambda_n, & \text{elsewhere.} \end{cases}$$

Here the parameter δ_n is equal to $-p(\mathfrak{v}_n)$, so that $\delta_n \to 0$, as $n \to \infty$, while the point $a_n \in \mathbb{R}^2$ is chosen large enough such that the supports of $\mathfrak{v}_n - \lambda_n$ and $w_{\delta_n}^n(\cdot - a_n) - \lambda_n$ do not intersect. We note that v_n belongs to $\mathcal{E}(\mathbb{R}^2)$, with $v_n - \lambda_n \in C_0^\infty(\mathbb{R}^2)$. Moreover, we have

$$d(v_n) = d(\mathfrak{v}_n) + d(w_{\delta_n}^n) = 0,$$

while, by construction,

$$|E(v_n) - E(v)| \leq |E(\mathfrak{v}_n) - E(v)| + |E(w_{\delta_n}^n)| \to 0,\ \text{and}\ d_{\mathcal{E}}^A(v_n, v) \leq d_{\mathcal{E}}^A(\mathfrak{v}_n, v) + d_{\mathcal{E}}^A(w_{\delta_n}^n, \lambda_n) \to 0,$$

as $n \to \infty$. For the momentum, using (5.3.6) we have

$$p(v_n) = p(\mathfrak{v}_n) - \int_{\mathbb{R}^2} x_2 w(w_{\delta_n}^n)(x - a_n)dx = p(\mathfrak{v}_n) + p(w_{\delta_n}^n) - 4\pi(a_n)_2 d(w_{\delta_n}^n).$$

Hence, since $p(w_{\delta_n}^n) = -p(\mathfrak{v}_n)$, $p(v_n) = 0$, which completes the proof of the first assertion.

The second assertion is then a direct consequence of the first one, once it is proved that the set

$$\{v \in \tilde{\mathcal{E}}(\mathbb{R}^2) : d(v) = 0,\ p(v) = \mathfrak{p}\}$$

is not empty. This again follows from Lemma 5.8.1. □

The previous results will allow us to deduce the Lipschitz continuity of the curve $E^0_{\min}$.

Corollary 5.8.3. *Let* $(\mathfrak{p},\mathfrak{q}) \in \mathbb{R}^2_+$. *Then,*

$$|E^0_{\min}(\mathfrak{p}) - E^0_{\min}(\mathfrak{q})| \leq |\mathfrak{p} - \mathfrak{q}|. \tag{5.8.9}$$

In particular, the function $\mathfrak{p} \mapsto E^0_{\min}(\mathfrak{p})$ *is continuous on* $\mathbb{R}_+$, *while the function* $p \mapsto \Xi(\mathfrak{p})$ *is nonnegative, nondecreasing and continuous on* $\mathbb{R}_+$.

Proof. We assume without loss of generality that $\mathfrak{q} \geq \mathfrak{p}$. We show first that

$$E_{\min}(\mathfrak{q}) \leq E_{\min}(\mathfrak{p}) + (\mathfrak{q} - \mathfrak{p}). \tag{5.8.10}$$

For that purpose, let $\delta > 0$ be given. By Lemma 5.8.2, there is a map $\mathfrak{v}_\delta \in \mathcal{E}^\infty_0(\mathbb{R}^2)$ and $\lambda_\delta \in \mathbb{S}^1 \times \{0\}$, with $\mathfrak{v}_\delta - \lambda_\delta \in C^\infty_0(\mathbb{R}^N)$, such that

$$p(\mathfrak{v}_\delta) = \mathfrak{p} \quad \text{and} \quad E(v_\delta) \leq E^0_{\min}(\mathfrak{p}) + \frac{\delta}{2}. \tag{5.8.11}$$

Now we set $\mathfrak{s} = \mathfrak{q} - \mathfrak{p}$ and invoking Lemma 5.8.1, we have a function $w_\delta \in \mathcal{E}^\infty_0(\mathbb{R}^2)$ such that $w_\delta - \lambda_\delta \in C^\infty_0(\mathbb{R}^2)$,

$$p(w_\delta) = \mathfrak{s} \quad \text{and} \quad E(w_\delta) \leq \mathfrak{s} + \frac{\delta}{2}. \tag{5.8.12}$$

Then we can take a sequence of points $a_\delta \in \mathbb{R}^2$ such that the supports of $w_\delta - \lambda_\delta$ and $\mathfrak{v}_\delta(\cdot - a_\delta) - \lambda_\delta$ do not intersect and then we define

$$v_\delta = \begin{cases} \mathfrak{v}_\delta, & \text{on } \operatorname{supp}(\mathfrak{v}_\delta - \lambda_\delta), \\ w_{\delta_\delta}(\cdot - a_\delta), & \text{on } \operatorname{supp}(w_\delta(\cdot - a_\delta) - \lambda_\delta), \\ \lambda_\delta, & \text{elsewhere}. \end{cases} \tag{5.8.13}$$

In particular, we have

$$E(v_\delta) = E(\mathfrak{v}_\delta) + E(w_\delta) \quad \text{and} \quad p(v) = p(\mathfrak{v}_\delta) + p(w_\delta) = \mathfrak{p} + \mathfrak{s} = \mathfrak{q}.$$

Then from (5.8.11) and (5.8.12) it follows that

$$E_{\min}(\mathfrak{q}) \leq E(v) = E(\mathfrak{v}_\delta) + \mathfrak{s} + \frac{\delta}{2} \leq E^0_{\min}(\mathfrak{p}) + (\mathfrak{q} - \mathfrak{p}) + \delta,$$

which yields (5.8.10) in the limit $\delta \to 0$.

Next we turn to the inequality

$$E_{\min}(\mathfrak{p}) \leq E_{\min}(\mathfrak{q}) + (\mathfrak{q} - \mathfrak{p}). \tag{5.8.14}$$

As before, in the proof of Lemma 5.8.2, we can construct functions $\tilde{\mathfrak{v}}_\delta, \tilde{w}_\delta \in \mathcal{E}^\infty_0(\mathbb{R}^2)$, such that $\tilde{\mathfrak{v}}_\delta - \tilde{\lambda}_\delta, \tilde{w}_\delta - \tilde{\lambda}_\delta \in C^\infty_0(\mathbb{R}^2)$, for some $\tilde{\lambda}_\delta \in \mathbb{S}^1 \times \{0\}$,

$$p(\tilde{\mathfrak{v}}_\delta) = \mathfrak{q}, \quad p(\tilde{w}_\delta) = -\mathfrak{s}, \quad E(\tilde{\mathfrak{v}}_\delta) \leq E_{\min}(\mathfrak{q}) + \frac{\delta}{2} \quad \text{and} \quad E(\tilde{w}_\delta) \leq \mathfrak{s} + \frac{\delta}{2}.$$

Then, considering a sequence of points $\tilde{a}_\delta \in \mathbb{R}^2$ such that the supports of $\tilde{w}_\delta - \tilde{\lambda}_\delta$ and $\tilde{\mathfrak{v}}_\delta(\cdot - \tilde{a}_\delta) - \tilde{\lambda}_\delta$ do not intersect, we define the function $\tilde{v}_\delta$ as in (5.8.13), changing $\mathfrak{v}_\delta$, w_δ and a_δ by $\tilde{\mathfrak{v}}_\delta$, $\tilde{w}_\delta$ and $\tilde{a}_\delta$, respectively. Then we deduce that $p(\tilde{v}_\delta) = p(\tilde{\mathfrak{v}}_\delta) - \mathfrak{s} = \mathfrak{p}$ and

$$E_{\min}(\mathfrak{p}) \leq E(\tilde{v}_\delta) = E(\tilde{\mathfrak{v}}_\delta) + E(\tilde{w}_\delta) \leq E_{\min}(\mathfrak{q}) + \mathfrak{s} + \delta,$$

so that (5.8.14) follows letting $\delta \to 0$. This completes the proof of Corollary 5.8.3, the last assertion being a consequence of (5.8.9). □

Lemma 5.8.4. *Let $\mathfrak{p}, \mathfrak{q} \geq 0$. Then,*

$$E^0_{\min}\Big(\frac{\mathfrak{p}+\mathfrak{q}}{2}\Big) \geq \frac{E^0_{\min}(\mathfrak{p}) + E^0_{\min}(\mathfrak{q})}{2}.$$

Proof. The main idea is to construct comparison maps using a reflection argument. For that purpose, for any $a \in \mathbb{R}$ and $v \in \tilde{\mathcal{E}}^\infty_0(\mathbb{R}^2)$, such that $d(v) = 0$, we consider the map $T^\pm_a v$ defined by $T^\pm_a v = v \circ P^\pm_a$, where P^+_a (resp. P^-_a) restricted to the set $\Gamma^+_a = \{x = (x_1, x_2) \in \mathbb{R}^2, x_2 \geq a\}$ (resp. the set $\Gamma^-_a = \{x = (x_1, x_2) \in \mathbb{R}^2, x_2 \leq a\}$) is the identity, whereas its restriction to the set Γ^-_a (resp. Γ^+_a) is the symmetry with respect to the plane of equation $x_2 = a$. In coordinates, this reads

$$T^+_a v(x_1, x_2) = \begin{cases} v(x_1, x_2), & \text{if } x_2 \geq a, \\ v(x_1, 2a - x_2), & \text{if } x_2 \leq a. \end{cases}$$

The expression for $T^-_a v$ is similar, reversing the inequalities. We verify that $T^\pm_a v$ belongs to $\tilde{\mathcal{E}}(\mathbb{R}^2)$ and that

$$E(T^\pm_a v) = 2 \int_{\Gamma^\pm_a} e(v). \tag{5.8.15}$$

Also, since $(T^\pm_a v)_3$ has compact support, using (5.3.6) we can compute

$$p(T^\pm_a v) = 2\left(- \int_{\Gamma^\pm_a} x_2 w(v) + a \int_{\Gamma^\pm_a} w(v) \right).$$

In particular, since $d(v) = 0$, this implies that

$$p(T^+_a v) + p(T^-_a v) = 2p(v). \tag{5.8.16}$$

We notice that the function $a \mapsto p(T^+_a f)$ is continuous and, by the Lebesgue theorem, tends to zero, as $a \to +\infty$. Also, since $d(v) = 0$, we have

$$p(T^+_a v) = 2\,(p(v) + ad(v)) = 2p(v), \quad \text{as } a \to -\infty.$$

Therefore, it follows by continuity that for every $\alpha \in (0, p(v))$, there exists a number $a \in \mathbb{R}$ such that

$$p(T^+_a v) = 2\alpha. \tag{5.8.17}$$

Hence, from (5.8.16)

$$p(T^-_a v) = 2(p(v) - \alpha). \tag{5.8.18}$$

Next, by Lemma 5.8.1, for any $\mathfrak{p}, \mathfrak{q} \geq 0$ and $\delta > 0$ there exists a function $w \in \tilde{\mathcal{E}}^\infty_0(\mathbb{R}^2)$ such that $d(w) = 0$,

$$p(w) = \frac{\mathfrak{p}+\mathfrak{q}}{2} \quad \text{and} \quad E(w) \leq E_{\min}\Big(\frac{\mathfrak{p}+\mathfrak{q}}{2}\Big) + \frac{\delta}{2}.$$

Invoking (5.8.17) and (5.8.18) for $v = w$ and $\alpha = p/2$, we can find some $a \in \mathbb{R}$ such that

$$p(T^+_a w) = \mathfrak{p} \quad \text{and} \quad p(T^-_a w) = \mathfrak{q}.$$

It then follows from (5.8.15) that

$$E_{\min}(\mathfrak{p}) \leq E(T^+_a w) \leq 2 \int_{\Gamma^+_a} e(w) \quad \text{and} \quad E_{\min}(\mathfrak{q}) \leq E(T^-_a w) \leq 2 \int_{\Gamma^-_a} e(w).$$

Adding these two inequalities, we obtain

$$E_{\min}(\mathfrak{p}) + E_{\min}(\mathfrak{q}) \leq 2E(w) \leq 2E_{\min}\Big(\frac{\mathfrak{p}+\mathfrak{q}}{2}\Big) + \delta.$$

The conclusion follows letting $\delta \to 0$. $\square$

Corollary 5.8.5. *The function $\mathfrak{p} \mapsto E^0_{\min}(\mathfrak{p})$ is concave and nondecreasing on $\mathbb{R}_+$.*

Proof. Continuous functions f satisfying the inequality

$$f\Big(\frac{\mathfrak{p}+\mathfrak{q}}{2}\Big) \geq \frac{f(\mathfrak{p}) + f(\mathfrak{q})}{2}$$

are concave. Similarly, concave nonnegative functions on $\mathbb{R}_+$ are nondecreasing, so that, in view of Corollary 5.8.3 and Lemma 5.8.4, $E^0_{\min}$ is concave and nondecreasing on $\mathbb{R}_+$. $\square$

Proof of Theorem 5.1.1. All the statements in Theorem 5.1.1 follows from Corollaries 5.8.3 and 5.8.5. $\square$

Proof of Proposition 5.1.3. For simplicity, we denote $u = u_{\mathfrak{p}}$. For all $\phi \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^3)$, we define the functionals dp and dE by

$$dE(u)[\phi] := \int_{\mathbb{R}^2} \nabla u : \nabla \phi_u^T + u_3\langle \phi_u^T, e_3\rangle,$$
$$dp(u)[\phi] := \int_{\mathbb{R}^2} \phi_u^T(u \times \partial_1 u) = \int_{\mathbb{R}^2} \phi(u \times \partial_1 u).$$

Now we claim that there exists $\varphi \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^3)$ such that

$$dp(u)[\varphi] \neq 0. \tag{5.8.19}$$

Indeed, otherwise $dp(u)[\phi] = 0$ for all $\phi \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^3)$, so that $u \times \partial_1 u = 0$ a.e. on $\mathbb{R}^2$. Then

$$x_2\partial_2 u \cdot (u \times \partial_1 u) = x_2 u \cdot (\partial_1 u \times \partial_2 u) = 0.$$

By definition of p, this gives that $p(u) = L(w(u)) = L(0) = 0$, which contradicts the fact that $p(u) = \mathfrak{p} > 0$. This yields (5.8.19).

Let $\psi \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^3)$ such that $dp(u)[\psi] = 0$. Then we define the function

$$F(s,t) = p(u + s\psi + t\varphi), \text{ for all } (s,t) \in \mathbb{R}^2,$$

so that $F(0,0) = \mathfrak{p}$,

$$\frac{\partial F}{\partial s}(0,0) = dp(u)(\psi) = 0 \quad \text{and} \quad \frac{\partial F}{\partial t}(0,0) = dp(u)(\varphi) \neq 0. \tag{5.8.20}$$

Therefore, by the implicit function theorem, there exist $\delta > 0$ and a function $f \in C^1((-\delta,\delta))$ such that $f(0) = f'(0) = 0$ and

$$p(u + s\psi + f(s)\varphi) = \mathfrak{p}, \text{ for all } s \in (-\delta,\delta).$$

Taking $\delta > 0$ smaller if necessary, since $d(u) = 0$, invoking Lemma B.6, we also have

$$d(u + s\psi + f(s)\varphi) = 0, \text{ for all } s \in (-\delta, \delta).$$

Therefore, by definition of u,

$$E(u) \leq E(u + s\psi + f(s)\varphi), \text{ for all } s \in (-\delta, \delta),$$

so that, differentiating at $s = 0$ and using that $f'(0) = 0$, we obtain $dE(u)(\psi) = 0$. Therefore (see e.g. [18, Lemma 3.2]) there exists a constant $c \in \mathbb{R}$ such that

$$c\, dp(u)[\phi] = dE(u)[\phi], \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^3). \tag{5.8.21}$$

Hence u is a solution of (TW$_c$) and Proposition 5.1.5 implies that u is smooth.

It only remains to prove (5.1.12). For this purpose, in view of (5.8.19), we consider

$$\tilde{\varphi} = \frac{\varphi}{dp(u)[\varphi]}.$$

Then, by (5.8.21), $c = dE(u)[\tilde{\varphi}]$ and defining the curve $\gamma := u_{\mathfrak{p}} + t\tilde{\varphi}$, we have

$$E(\gamma(t)) = E_{\min}(\mathfrak{p}) + ct + \underset{t \to 0}{O}(t^2).$$

Moreover, from (5.3.31),

$$p(\gamma(t)) = \mathfrak{p} + \mathfrak{s}, \text{ where } \mathfrak{s} = t + t^2 K(t, u, \tilde{\varphi}),$$

and $K(t, u, \tilde{\varphi})$ is C^1-function of t, for t small. Then, by the implicit function theorem, in a neighborhood of the origin we can express the relation $\mathfrak{s} = t + t^2 K(t, u, \tilde{\varphi})$ as a function of $\mathfrak{s}$, that is

$$t(\mathfrak{s}) = \mathfrak{s} + o(\mathfrak{s}^2).$$

Consequently,

$$E_{\min}(\mathfrak{p} + \mathfrak{s}) - E_{\min}(\mathfrak{p}) \leq E(\gamma(t(\mathfrak{s}))) - E_{\min}(\mathfrak{p}) \leq c\mathfrak{s} + \underset{\mathfrak{s} \to 0}{O}(\mathfrak{s}^2),$$

and letting $\mathfrak{s} \to 0^{\pm}$,

$$\frac{d^+}{dp}\big(E_{\min}(p)\big) \leq c(u_p) \leq \frac{d^-}{dp}\big(E_{\min}(p)\big).$$

By combining with (5.1.11), (5.1.12) follows. If $c = 1$, since $E(u) = E_{\min}(\mathfrak{p}) \leq \mathfrak{p} = p(u)$, by (5.4.2) we conclude that

$$\int_{\mathbb{R}^2} |\partial_2 u|^2 \leq 0,$$

which implies that u is constant, in contradiction with the fact that $\mathfrak{p} > 0$. Finally, we rule out that $c = 0$ by Proposition 5.1.8. □

Finally, we proof that the infimum of $E^0_{\min}(\mathfrak{p})$ is not attained for $\mathfrak{p} > 0$ small.

Proof of Theorem 5.1.10. Arguing by contradiction, we assume that for some $\mathfrak{p} \in (0, \kappa_0)$, there exists $u_{\mathfrak{p}} \in \tilde{\mathcal{E}}(\mathbb{R}^2)$ such that

$$E^0_{\min}(\mathfrak{p}) = E(u_{\mathfrak{p}}).$$

Since by Theorem 5.1.1 we have $E^0_{\min}(\mathfrak{p}) \leq \mathfrak{p}$, we conclude that $E(u_{\mathfrak{p}}) < \kappa_0$, which is a contradiction with Theorem 5.1.9. □

5.9 The one-dimensional case

In this section we consider the case $N = 1$. Then the equation (TW$_c$) is integrable and the solutions are given explicitly by the next proposition.

Proposition 5.9.1 ([79, 81, 91]). *Let $N = 1$ and $c \geq 0$. Assume that $u \in \mathcal{E}(\mathbb{R})$ is a nontrivial solution of* (TW$_c$). *Then $0 \leq c < 1$ and, up to a translation of u and a rotation of $\check{u}$, the solution is given by*

$$u_1 = c\,\mathrm{sech}(\sqrt{1-c^2}\,x), \tag{5.9.1}$$

$$u_2 = \tanh(\sqrt{1-c^2}\,x), \tag{5.9.2}$$

$$u_3 = \pm\sqrt{1-c^2}\,\mathrm{sech}(\sqrt{1-c^2}\,x). \tag{5.9.3}$$

Moreover, if $0 < c < 1$, we can write

$$\check{u} = \sqrt{1-u_3^2}\exp(i\theta), \tag{5.9.4}$$

where

$$\theta = \arctan\left(\frac{\sinh(\sqrt{1-c^2}\,x)}{c}\right). \tag{5.9.5}$$

Proof. Since $N = 1$, (TW$_c$) reads

$$-u_1'' = 2e(u)u_1 + c(u_2u_3' - u_3u_2'), \tag{5.9.6}$$

$$-u_2'' = 2e(u)u_2 + c(u_3u_1' - u_1u_3'), \tag{5.9.7}$$

$$-u_3'' = 2e(u)u_3 - u_3 + c(u_1\partial_1u_2 - u_2u_1'). \tag{5.9.8}$$

Also, as in (5.5.6), we have

$$(u_1u_2' - u_1'u_2)' = cu_3'. \tag{5.9.9}$$

Thus, imposing that u_1', u_2' and u_3 vanish at infinity, we integrate (5.9.9) to obtain

$$u_1u_2' - u_1'u_2' = cu_3. \tag{5.9.10}$$

Replacing (5.9.9) in (5.9.8), we get

$$u_3'' + 2e(u)u_3 - (1-c^2)u_3 = 0. \tag{5.9.11}$$

Now, multiplying (5.9.6), (5.9.7), (5.9.11) by u_1', u_2', u_3', respectively and adding these relations we have

$$-(|u'|^2)' = 2e(u)(u_1^2 + u_2^2 + u_3^2)' - c^2(u_3^2)'. \tag{5.9.12}$$

Since $|u| = 1$, $(u_1^2 + u_2^2 + u_3^2)' = 0$. Therefore integrating (5.9.12) we conclude that

$$|u'|^2 = u_3^2, \tag{5.9.13}$$

so that $e(u) = u_3^2$ and equation (5.9.11) reduces to

$$u_3'' - 2u_3^3 - (1-c^2)u_3 = 0. \tag{5.9.14}$$

As before, multiplying (5.9.14) by u_3' and integrating, we conclude that

$$(u_3')^2 = u_3^2((1-c^2) - u_3^2). \tag{5.9.15}$$

Since equation (5.9.14) is invariant under translation, supposing u_3 not identically zero, we can assume that

$$|u_3(0)| = \max\{|u_3(x)| : x \in \mathbb{R}\} > 0.$$

Therefore

$$u_3'(0) = 0, \tag{5.9.16}$$

and from (5.9.15) and (5.9.16),

$$u_3^2(0) = 1 - c^2. \tag{5.9.17}$$

In particular we deduce that if $c \geq 1$, $u_3 \equiv 0$, which implies that u_1 and u_2 are constant. If $0 \leq c < 1$, by the Cauchy–Lipschitz theorem, equation (5.9.14) with initial conditions (5.9.16) and $u_3(0) = \sqrt{1-c^2}$ or $u_3(0) = -\sqrt{1-c^2}$ has a unique solution. It is straightforward to check that

$$u_3(x) = \pm\sqrt{1-c^2}\,\operatorname{sech}(\sqrt{1-c^2}\,x) \tag{5.9.18}$$

is the desired solution. Moreover, (5.9.18) shows that $\|u_3\|_{L^\infty(\mathbb{R})} < 1$ if $c \in (0,1)$. Hence, for $c \in (0,1)$, we can write

$$\check{u} = \sqrt{1-u_3^2}e^{i\theta},$$

and then (5.9.9) yields

$$\theta' = \frac{cu_3}{1-u_3^2}. \tag{5.9.19}$$

From (5.9.18) and (5.9.19), we are led to

$$\theta = \theta_0 + \arctan\left(\frac{\sinh(\sqrt{1-c^2}\,x)}{c}\right),$$

for some constant $\theta_0 \in \mathbb{R}$, which proves (5.9.4)–(5.9.5). Using some standard identities for trigonometric and hyperbolic functions, we also obtain (5.9.1)–(5.9.3), for $c \in (0,1)$. It only remains to show that for $c = 0$, (5.9.1) and (5.9.2) are the unique solutions of (5.9.6)–(5.9.7). Indeed, since $e(u)(x) = u_3^2(x) = \operatorname{sech}^2(x)$, we recast (5.9.1) and (5.9.2) as

$$-\check{u}'' = 2\operatorname{sech}^2(x)\check{u}, \tag{5.9.20}$$

and from (5.9.13) we can asumme that, up to a multiplication by a complex number of modulus one,

$$\check{u}'(0) = 1. \tag{5.9.21}$$

Then the Cauchy–Lipschitz theorem provides the existence of a unique solution of (5.9.20)–(5.9.21) in a neighborhood of $x = 0$, and it is immediate to check that $\check{u}(x) = \tanh(x)$ is the solution, which concludes the proof. □

In the one-dimensional case, the momentum is formally given by

$$p(u) = \int_{\mathbb{R}} \frac{u_3(u_1u_2' - u_2u_1')}{1-u_3^2}.$$

If $\|u_3\|_{L^\infty(\mathbb{R})} < 1$, we see that

$$p(u) = \int u_3\theta',$$

and therefore it agrees with the corresponding expression in the higher dimensional case. Now we have the following.

Corollary 5.9.2. *Assume that $c \in [0,1)$ and let $u_c \in \mathcal{E}(\mathbb{R})$ be a solution of* (TW_c). *Then*

$$E(u_c) = 2\sqrt{1-c^2}. \tag{5.9.22}$$

Moreover,

$$p(u_c) \equiv \int_{\mathbb{R}} u_3\theta' = 2\arctan\left(\frac{\sqrt{1-c^2}}{c}\right), \ \textit{for } c \in (0,1). \tag{5.9.23}$$

In particular, we can write explicitly E as a function of p as

$$E(p) = 2\sin(p/2) \tag{5.9.24}$$

and

$$\frac{dE}{dp} = \cos(p) = c, \tag{5.9.25}$$

for $c \in (0,1)$.

Proof. Using (5.9.3) and (5.9.13), we have

$$E(u) = \int_{\mathbb{R}} u_3^2 = \sqrt{1-c^2}\int_{\mathbb{R}} \operatorname{sech}^2(x)\,dx = 2\sqrt{1-c^2}.$$

For the momentum, (5.9.19) yields

$$p(u) = \int_{\mathbb{R}} u_3\theta' = c\int_{\mathbb{R}} \frac{u_3^2}{1-u_3^2} = c(1-c^2)\int_{\mathbb{R}} \frac{\operatorname{sech}^2(\sqrt{1-c^2}x)}{1-(1-c^2)\operatorname{sech}^2(\sqrt{1-c^2}x)}dx.$$

Then, using the change of variables $y = \frac{\sqrt{1-c^2}}{c}\tanh(\sqrt{1-c^2}\,x)$, we obtain (5.9.23), from where we deduce that

$$c = \frac{1}{\tan^2(p/2)} = \cos(p/2). \tag{5.9.26}$$

Finally, from (5.9.22) and (5.9.26), we establish (5.9.24), from where (5.9.25) is an immediate consequence. □

5.10 Appendix

For the convenience of the reader we recall some well-known results used in this paper. We assume Ω to be a smooth open bounded domain of $\mathbb{R}^N$.

Theorem B.1 ([97, 67]). *Let $u \in H^1(\Omega)$, such that $\Delta u = 0$ on $D'(\Omega)$. Then there are constants $0 < \alpha \le 1$, $\alpha = \alpha(N)$, and $K > 0$ such that if $x \in \Omega$ and $0 < \rho < r < dist(x, \Omega)$,*

$$\operatorname*{osc}_{B_\rho} u \le K\left(\frac{\rho}{r}\right)^\alpha \frac{\|u\|_{L^2(B_r)}}{r^{N/2}}.$$

Moreover, if $N=2$, then

$$\operatorname*{osc}_{B_\rho} u \leq K(\ln(\rho/r))^{-1/2}\|\nabla u\|_{L^2(B_r)}.$$

for some $K>0$.

Theorem B.2 ([97]). *Let $p>N/2$ and $f\in L^p(\Omega)$. Assume that $u\in H_0^1(\Omega)$ is solution of*

$$-\Delta u = f, \qquad \text{in } \Omega.$$

Then u is Hölder continuous in $\bar{\Omega}$. Moreover, for $\rho>0$, there exists a constant $K(\rho)$ such that

$$\operatorname*{osc}_{B_\rho\cap\Omega} u \leq K(\rho)\|f\|_{L^p(\Omega)}.$$

Lemma B.3. *Let $f\in L^1(\mathbb{R}^N)$. Then for every $\varepsilon>0$ there exists $K(\varepsilon)$ such that $f=f_1+f_2$ a.e. on $\mathbb{R}^N$ and*

$$\|f_2\|_{L^1(\mathbb{R}^2)}\leq\varepsilon, \quad \|f_1\|_{L^\infty(\mathbb{R}^2)}\leq K(\varepsilon).$$

Proof. Let

$$f_{1,k}=\begin{cases} k, & \text{if } f\geq k,\\ f, & \text{if } |f|\leq k,\\ -k, & \text{if } f\leq -k.\end{cases}$$

and $f_{2,k}=f-f_{1,k}$. Then

$$\|f_{2,k}\|_{L^1(\mathbb{R}^N)}\leq 2\int_{\{|f|\geq k\}}|f|. \tag{B.1}$$

Since

$$|\{|f|\geq k\}|=\int_{\{|f|\geq k\}}1\leq\int_{\mathbb{R}^2}\frac{1}{k}\|f\|_{L^1(\mathbb{R}^2)}\to 0, \quad \text{as } k\to\infty,$$

invoking the dominated convergence theorem and (B.1), we conclude that $\|f_{2,k}\|_{L^1(\mathbb{R}^N)}\to 0$, as $k\to\infty$ and the conclusion follows. □

Lemma B.4. *Let $N\geq 1$. Assume that $f\in L^p(B(0,R_0)^c)$, for some $R_0\geq 0$ and $p\in[1,\infty)$. Let $q\geq 1$ such that $q\in[(1-1/N)p,p]$. Then there exists a sequence $R_n\to\infty$ such that for all $s\in[0,Nq/p-N+1]$ we have*

$$R_n^s\int_{\partial B(0,R_n)}|f|^q d\sigma\leq\frac{K(p,q,N)}{(\ln R_n)^{q/p}}, \quad \text{as } n\to\infty,$$

for some constant $K(p,q,N)>0$.

Proof. Since $f\in L^p(B(0,R_0)^c)$,

$$\int_{R_0}^\infty\left(\int_{\partial B(0,r)}|f|^p\right)dr<\infty,$$

and thus there is a sequence $R_n\to\infty$, as $n\to\infty$, such that

$$\int_{\partial B(0,R_n)}|f|^p\leq\frac{1}{R_n\ln(R_n)}.$$

Then, using the Hölder inequality we obtain

$$\left(\int_{\partial B(0,R_n)} |f|^q\right)^{1/q} \leq (C(N)R_n^{N-1})^{1/q-1/p}\frac{1}{(R_n \ln R_n)^{1/p}},$$

from where the result follows. □

Lemma B.5. *Let $c \geq 0$ and $u \in C^\infty(\mathbb{R}^N) \cap UC(\mathbb{R}^N)$ be a solution of* (TW$_c$)*. Assume that*

$$\operatorname*{osc}_{B(y,r)} u \leq \frac{1}{8(1+c)(2s+1)}, \tag{B.2}$$

for some $y \in \mathbb{R}^N$, $r > 0$ and $s \geq 1$. Then

$$\int_{B(y,r/2)} |\nabla u|^{2(s+1)} \leq 4(1+c)^2\left(1+\frac{16}{r^2}\right)\int_{B(y,r)} |\nabla u|^{2s}. \tag{B.3}$$

Proof. The ideas of the proof are based on classical computations for elliptic equations with quadratic growth (see e.g. [69, 16, 64]). Therefore we only provide the main ideas, in order to show the dependence on u, c, s and N as stated. We set $B_r \equiv B(y,r)$ and $\eta \in C_0^\infty(B_r)$ a function such that $0 \leq \eta \leq 1$,

$$|\nabla \eta| \leq \frac{4}{r} \text{ on } B_r \quad \text{and} \quad \eta \equiv 1 \text{ on } B_{r/2}. \tag{B.4}$$

Finally, we fix $w = |\nabla u|^2$, which is smooth by hypothesis, so that

$$|\nabla w| \leq 2w^{1/2}|D^2u|. \tag{B.5}$$

We now divide the computations in several steps.

Step 1. If $\operatorname{osc}_{B_r} u \leq 1/4$, we have

$$\int_{B_r} \eta^2 w^{s+1} \leq 2\operatorname*{osc}_{B_r} u\left(\int_{B_r} |\nabla\eta|^2 w^s + \frac{2s+1}{2}\int_{B_r} \eta^2|D^2u|w^{s-1}\right).$$

Indeed, since

$$\int_{B_r} \eta^2 w^{s+1} = \int_{B_r} \eta^2 \nabla u - u(y)\cdot \nabla u w^2,$$

integrating by parts and using (B.5), we deduce that

$$\int_{B_r} \eta^2 w^{s+1} \leq \operatorname*{osc}_{B_r} u\left(2\int_{B_r} \eta|\nabla\eta|w^{s+1/2} + (2s+1)\int_{B_r} \eta^2|D^2u|w^s\right). \tag{B.6}$$

Using the elementary inequalities $2ab \leq a^2 + b^2$ and $ab \leq a^2 + b^2/4$ in the first and second integrals in the r.h.s. of (B.6), we obtain

$$(1 - 2\operatorname*{osc}_{B_r} u)\int_{B_r} \eta^2 w^{s+1} \leq \operatorname*{osc}_{B_r} u\left(\int_{B_r} |\nabla\eta|^2 w^s + \frac{(2s+1)^2}{4}\int_{B_r} \eta^2|D^2u|w^{s-1}\right).$$

Since $\operatorname{osc}_{B_r} u \leq 1/4$, we conclude Step 1.

Step 2. We have

$$\frac{1}{2}\int_{B_r}\eta^2|D^2u|^2w^{s-1}\leq 2\int_{B_r}|\nabla\eta|^2w^s+\sum_{k=1}^N\int_{B_r}\partial_k(\Delta u)\cdot\partial_k u\,\eta^2w^{s-1}.$$

Let $k\in\{1,\dots,N\}$ and $\phi_k=\eta^2w^{s-1}\partial_k u\in C_0^\infty(B_r)$. Then, several integrations by parts yield

$$\sum_{j=1}^N\int_{B_r}\partial^2_{ki}u\cdot\phi_k=\int_{B_r}\partial_k(\Delta u)\cdot\phi_k. \tag{B.7}$$

On the other hand, using that

$$\partial_i w=2\sum_{j=1}^N\partial u_j\cdot\partial^2_{ki}u,$$

a straightforward computation gives

$$\begin{aligned}\sum_{j,k=1}^N\int_{B_r}\partial^2_{ki}u\cdot\phi_k=\int_{B_r}\eta^2|D^2u|^2w^{s-1}+2\sum_{j,k=1}^N\int_{B_r}\eta\partial_j\eta w^{s-1}\partial^2_{jk}u\cdot\partial_k u\\ +\frac{s-1}{2}\int_{B_r}\eta^2w^{s-2}|\nabla w|^2.\end{aligned} \tag{B.8}$$

Then the conclusion of this step follows combining (B.7) and (B.8), noticing that the last integral in the r.h.s. of (B.8) is nonnegative, and that

$$2\sum_{j,k=1}^N|\eta\partial_j\eta w^{s-1}\partial^2_{jk}u\cdot\partial_k u|\leq 2\sum_{j,k=1}^N\eta|\partial^2_{jk}u|w^{\frac{s-1}{2}}\cdot|\partial_j\eta||\partial_k u|w^{\frac{s-1}{2}}\leq\frac{1}{2}\eta^2|D^2u|^2w^{s-1}+2|\nabla\eta|^2w^s.$$

Step 3. For all $\delta>0$, we have

$$\sum_{j=1}^N|\partial_j\Delta u\cdot\partial_j u|\leq\delta(c+1)|D^2u|^2+\left(c+\frac{c}{\delta}+4\right)w+(1+c)w^2.$$

Using (TW$_c$) and the fact that $|u|=1$, it is simple to check that

$$\sum_{j=1}^N|\partial_j\Delta u\cdot\partial_j u|\leq 2w|D^2u|+w^2+4w+2cw^{3/2}+2c|D^2u|w^{1/2}.$$

Combining with the fact that $2ab\leq\delta a^2+\delta^{-1}b$, for all $\delta>0$, we finish Step 3.

Step 4.

$$\frac{1}{4}\int_{B_r}\eta^2|D^2u|^2w^{s-1}\leq 2\int_{B_r}|\nabla\eta|^2w^s+(4c^2+5c+4)\int_{B_r}\eta^2w^s+(c+1)\int_{B_r}\eta^2w^{s+1}.$$

Step 4 follows immediately from Steps 2 and 3, taking $\delta=(4(c+1))^{-1}$.

Now we are in position to finish the proof of Lemma B.5. In fact, combining Steps 1 and 4, we are led to

$$(1-4(c+1)(2s+1)\operatorname*{osc}_{B_r} u)\int_{B_r}\eta^2 w^{s+1} \le 2\operatorname*{osc}_{B_r} u\left((8s+5)\int_{B_r}|\nabla\eta|^2 w^s + \right. \\ \left. 2(4c^2+5c+4)(2s+1)\int_{B_r}\eta^2 w^s\right).$$

Also, we see that (B.2) implies that

$$1/2 \le (1-4(c+1)(2s+1)\operatorname*{osc}_{B_r} u),$$

and that

$$2\operatorname*{osc}_{B_r} u\cdot\max\{8s+5, 2(4c^2+5c+4)(2s+1)\} \le \frac{2(4c^2+5c+4)}{4(1+c)} \le 2(1+c^2).$$

By combining with (B.4), we conclude (B.3). □

Lemma B.6. *The degree given by* (5.1.3) *is well-defined and continuous from* $\mathcal{E}(\mathbb{R}^2)$ *to* $\mathbb{Z}$.

Proof. Since

$$|w(v)| \le |\partial_1 v||\partial_2 v| \le \frac{1}{2}|\nabla v|^2 \le e(v), \tag{B.9}$$

the functional d is well-defined from $\mathcal{E}(\mathbb{R}^2)$ to $\mathbb{R}$. Concerning the continuity, we know that, given $\varepsilon > 0$ and a $\mathfrak{v}\in\mathcal{E}(\mathbb{R}^2)$, there exists $R>0$ such that

$$\int_{B(0,R)^c} e(\mathfrak{v}) \le \varepsilon.$$

Invoking again (B.9), we deduce that

$$\Big|\int_{B(0,R)^c} w(w)\Big| \le \frac{1}{2}\int_{B(0,R)^c}|\nabla w|^2 \le \varepsilon + \frac{1}{2}\big(\|\nabla\mathfrak{v}\|_{L^2(\mathbb{R}^2)}+\|\nabla w\|_{L^2(\mathbb{R}^2)}\big)\|\nabla w-\nabla\mathfrak{v}\|_{L^2(\mathbb{R}^2)}, \tag{B.10}$$

for any $w\in\mathcal{E}(\mathbb{R}^2)$. We next introduce a function $\chi\in C_0^\infty(\mathbb{R}^2,[0,1])$ such that $\chi=1$ on $B(0,R)$ and $\chi=0$ on $B(0,2R)^c$. Combining with (B.9) and (B.10), we are led to

$$\big|d(\mathfrak{v})-d(w)\big| \le \frac{1}{4\pi}\Big|\int_{\mathbb{R}^2}\chi\big(w(\mathfrak{v})-w(w)\big)\Big| + \frac{1}{4\pi}\Big(\varepsilon+\frac{1}{2}\big(\|\nabla\mathfrak{v}\|_{L^2(\mathbb{R}^2)}+\|\nabla w\|_{L^2(\mathbb{R}^2)}\big)\|\nabla w-\nabla\mathfrak{v}\|_{L^2(\mathbb{R}^2)}\Big). \tag{B.11}$$

We also have

$$w(\mathfrak{v})-w(w) = \langle\mathfrak{v}-w,\partial_1\mathfrak{v}\times\partial_2\mathfrak{v}\rangle + \langle w,\partial_1(\mathfrak{v}-w)\times\partial_2\mathfrak{v}\rangle + \langle w,\partial_1 w\times\partial_2(\mathfrak{v}-w)\rangle, \tag{B.12}$$

so that

$$\Big|\int_{\mathbb{R}^2}\chi\big(w(\mathfrak{v})-w(w)\big)\Big| \le \Big|\int_{\mathbb{R}^2)}\chi\langle\mathfrak{v}-w,\partial_1\mathfrak{v}\times\partial_2\mathfrak{v}\rangle\Big| \\ + \big(\|\nabla\mathfrak{v}\|_{L^2(\mathbb{R}^2)}+\|\nabla w\|_{L^2(\mathbb{R}^2)}\big)\|\nabla\mathfrak{v}-\nabla w\|_{L^2(\mathbb{R}^2)}. \tag{B.13}$$

Integrating by parts the integral in the r.h.s. of (B.13), we compute

$$\begin{aligned}\int_{\mathbb{R}^2} \chi\langle \mathfrak{v}-w, \partial_1\mathfrak{v}\times\partial_2\mathfrak{v}\rangle =& \frac{1}{2}\Big(- \int_{\mathbb{R}^2} \partial_1\chi\langle \mathfrak{v}-w, \mathfrak{v}\times\partial_2\mathfrak{v}\rangle - \int_{\mathbb{R}^2} \chi\langle \partial_1(\mathfrak{v}-w), \mathfrak{v}\times\partial_2\mathfrak{v}\rangle \\ &+ \int_{\mathbb{R}^2} \partial_2\chi\langle \mathfrak{v}-w, \mathfrak{v}\times\partial_1\mathfrak{v}\rangle + \int_{\mathbb{R}^2} \langle \partial_2(\mathfrak{v}-w), \mathfrak{v}\times\partial_1\mathfrak{v}\rangle\Big).\end{aligned} \tag{B.14}$$

Actually, the integration by parts is only possible when the derivative $\partial_{12}^2\mathfrak{v}$ makes sense and is sufficiently integrable. However, we can use the density of $H^2(\mathbb{R}^2;\mathbb{R}^3)$ into $H^1(\mathbb{R}^2;\mathbb{R}^3)$ to approach the function $\mathfrak{v}$ by a sequence of functions $(\mathfrak{v}_n)_{n\in\mathbb{N}}$ for which (B.14) is satisfied. Then, we will invoke the continuity of both the left and right-hand sides of (B.14) with respect to the convergence in $H^1(\mathbb{R}^2,\mathbb{R}^3)$ in order to establish (B.14) in the general case. In any case, we deduce from (B.14) that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} \chi\langle \mathfrak{v}-w, \partial_1\mathfrak{v}\times\partial_2\mathfrak{v}\rangle\Big| \leq& \|\nabla\chi\|_{L^\infty(\mathbb{R}^2)}\|\mathfrak{v}-w\|_{L^2(B(0,2R))}\|\nabla\mathfrak{v}\|_{L^2(\mathbb{R}^2)} \\ &+ \|\nabla\mathfrak{v}-\nabla w\|_{L^2(\mathbb{R}^2)}\|\nabla\mathfrak{v}\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Combining with (B.11) and (B.13), we are led to

$$\big|d(\mathfrak{v})-d(w)\big| \leq C\Big(\|\nabla\mathfrak{v}\|_{L^2(\mathbb{R}^2)}+\|\nabla w\|_{L^2(\mathbb{R}^2)}\Big)\Big(\|\nabla\mathfrak{v}-\nabla w\|_{L^2(\mathbb{R}^2)}+\|\mathfrak{v}-w\|_{L^2(B(0,2R))}\Big)+\varepsilon, \tag{B.15}$$

where $C = \max\{3, \|\nabla\chi\|_{L^\infty(\mathbb{R}^2)}\}/4\pi$. At this stage, recall that, given any positive number A, there exists a positive constant $K(A,R)$ such that

$$d_{\mathcal{E}}^{2R}(f,g) \leq K(A,R) d_{\mathcal{E}}^A(f,g),$$

for any $f,g\in\mathcal{E}(\mathbb{R}^2)$. In view of (B.15), it follows that the functional d is continuous on $\mathcal{E}(\mathbb{R}^2)$. In particular, d is integer-valued since its restriction to the dense subset $\mathcal{E}^\infty(\mathbb{R}^2)$ is integer-valued. This completes the proof of Lemma B.6. □

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